Signal Reconstruction from Sparse Representations: An Introduction to Compressed Sensing

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Digital Data Acquisition

Suppose we want to acquire some real world signal digitally. Applications abound:

- Digital cameras
- Medical imaging
- Sound recording
- Temperature measurements
- etc...

How many measurements do we need to take?

Digital Data Acquisition

- Classical answer: Nyquist-Shannon theorem. Sampling rate must be at least the Nyquist frequency $2W$

This works for *bandlimited* signals. We can get by with fewer measurements if we know signals of interest are *sparse* or *compressible*; this is Compressive Sensing.

Linear Measurements

In CS we model the acquisition process by *linear measurements*. Given a measurement matrix $\Phi \in \mathbb{R}^{m \times n}$, form measurement

$$y = \Phi x.$$

This is the only information about x that we can access.

The Basic Problem

How do we reconstruct a vector x given only the measurement

$$y = \Phi x?$$

Assume $\Phi \in \mathbb{R}^{m \times n}$ is full rank. Then either:

- $m \geq n$: system is *fully determined*

or

- $m < n$: system is *underdetermined*.

Overdetermined Case

If $\Phi \in \mathbb{R}^{m \times n}$ is full rank with $m \geq n$, unique solution to

$$y = \Phi x$$

is

$$x = (\Phi^* \Phi)^{-1} \Phi^* y.$$

If the number of measurements is at least the length of x then we can reconstruct x exactly.

Underdetermined Case

More interesting: what if $\Phi \in \mathbb{R}^{m \times n}$ and $m < n$?

$$y = \Phi x$$

has infinitely many solutions. $null(\Phi)$ is nontrivial, and

$$\Phi x = \Phi(x + n), \forall n \in null(\Phi).$$

Can determine reconstruction only up to affine space

$$x + null(\Phi).$$

Least squares solutions

How do we pick the "best" solution in the affine space $x + null(\Phi)$? Simplest method: take solution with minimum l^2 norm (least squares)

$$x = \Phi^{+} y = \Phi^{*}(\Phi\Phi^{*})^{-1} y$$

Least squares solutions

Least squares is fast, reliable, but often gives poor reconstructions.

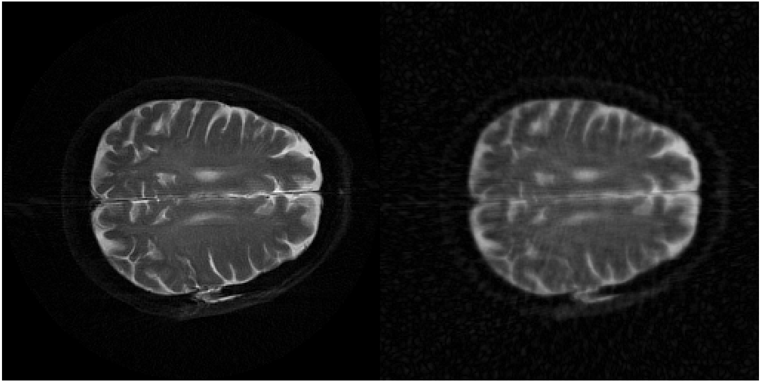


Figure: (left) original image, (right) least squares reconstruction

Better reconstructions

To improve on least squares, need more info about signals of interest. In CS we assume signals are *sparse* or *compressible* under some transform Ψ.

- Sparse: Ψx has only a few nonzero coefficients
- Compressible: well-approximated by sparse signal

l^0 minimization

We look for the *sparsest* solution in affine space $x + null(\Phi)$; solve

$$argmin_x ||\Psi x||_0$$

subject to

$$\Phi x = y$$

Uniqueness of sparse reconstruction

Theorem

If x is s-sparse and any 2s columns of A are linearly independent, then x is the unique s-sparse solution to $y = Ax$.

Proof.

Suppose $Ax = Az$. Then $A(x - z) = 0$, and $x - z$ is 2s-sparse. Since any 2s columns of A are linearly independent, this means $x - z = 0$. □

Uniqueness of sparse reconstruction

So, if any 2s columns $\Phi\Psi^{-1}$ are linearly independent, then there is a unique solution to the l^0 minimization problem for any s-sparse x. We can reconstruct sparse signals from far fewer than the full set of measurements this way!

Example

$\Phi \in \mathbb{R}^{4\times 10}$ with columns taken from the 10×10 Fourier matrix satisfies the conditions of the theorem. Any 2-sparse signal in $\mathbb{R}^{10}$ can be recovered from just 4 frequency measurements.

Problems with l^0 minimization

Unfortunately, l^0 minimization is a non-convex optimization problem and is intractable in practice (NP hard in general). We need alternate methods to find sparse solutions.

- l^2 minimization: fast and reliable, but inaccurate for sparse signals
- l^0 minimization: optimal reconstructions for sparse signals, usually can't be solved in practice.

Other Solution Methods

- l^1 minimization
- Orthogonal Matching Pursuit
- Iterative Hard Thresholding

Orthogonal Matching Pursuit

Orthogonal Matching Pursuit, or OMP: build sparse representation one basis vector at a time. Given $y = \Phi x$, look for an s-sparse solution α

- Start with $\alpha = 0$.
- At each step, compute residual $r_k = y - \Phi\alpha$
- Choose column of Φ that is most correlated with r_k and update α

Iterative Hard Thresholding

Much faster than OMP; only works for very sparse signals.

$$\alpha_{n+1} = H_s(\alpha_n + \Psi\Phi^*(y - \Phi\Psi^*\alpha_n))$$

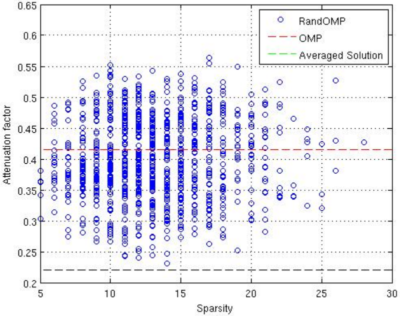
H_s: hard thresholding operator, keep only s largest coefficients

Fusing multiple representations

If we can find multiple sparse solutions to $y = \Phi x$, we can often combine them into a better solution, e.g. by averaging.

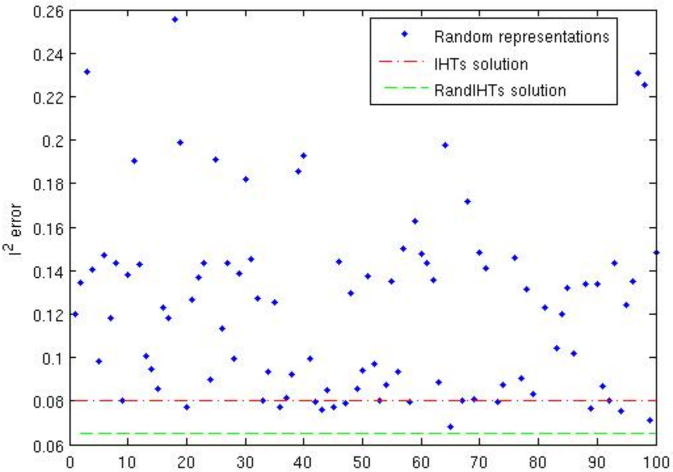
RandOMP: Improving the OMP Solution

RandOMP randomizes OMP algorithm. At each step, update support of α randomly. Run RandOMP multiple times, take average of the sparse solutions we find.

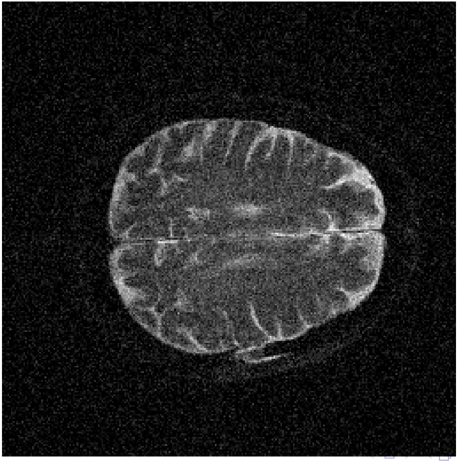


Randomized IHTs

Can apply same idea as RandOMP to IHTs. Use randomized thresholding operator $RandH_s$; choose large coefficients to keep from a distribution.


l^2 error
0.26
0.24
0.22
0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0
10
20
30
40
50
60
70
80
90
100
Random representations
IHTs solution
RandIHTs solution

Randomized IHTs Reconstruction of a Very Noisy Image



Randomized IHTs Reconstruction of a Very Noisy Image

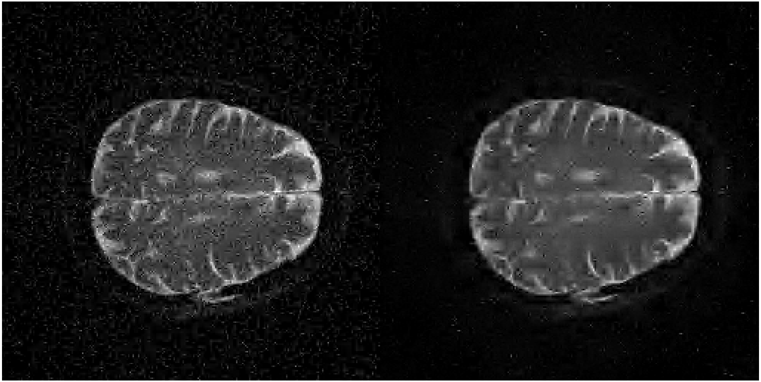


Figure: (left) IHTs, (right) RandIHTs Averaged Solutions

The End

Thanks to Dr. Ali Bilgin for advising this project.

Primary references:

- *An introduction to compressive sampling*, E. Candes and M. Wakin
- *A Plurality of Sparse Representations is Better than the Sparsest One Alone*, M. Elad and I. Yavneh
- *Robust uncertainty principles: Exact signal reconstruction from incomplete frequency information*, E. Candes, J Romberg, and T. Tao
- *Compressed sensing*, D. Donoho
- *Iterative Hard Thresholding for Compressed Sensing*, T. Blumensath and M. Davies

Coherence

Coherence between orthonormal bases Φ, Ψ is

$$\mu(\Phi, \Psi) = \sqrt{n} \cdot max(\phi_i, \psi_j)$$

Satisfies $\mu \in [1, \sqrt{n}]$. $\mu = 1$ is *maximal incoherence*. Want sparsity and measurement bases to be as incoherent as possible.

Low coherence pairs

In compressed sensing we want the measurement/sparsity bases to be as incoherent as possible. Some examples of low-coherence pairs:

- Standard basis and Fourier basis; maximally incoherent
- Fourier and wavelet bases; largely incoherent
- Random bases are incoherent with any fixed basis with high probability

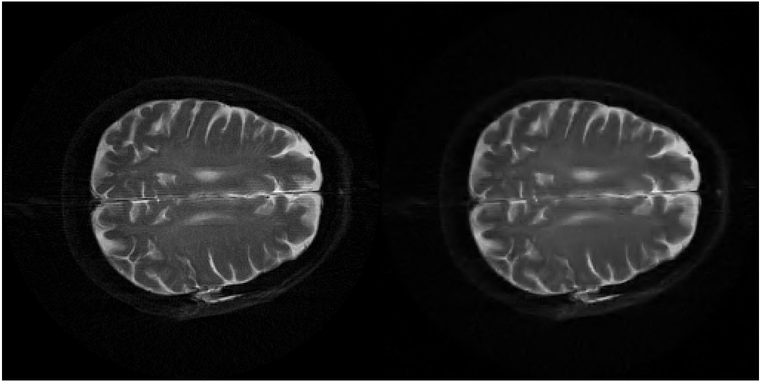


Figure: (left) original image, (right) sparse reconstruction