Signal Reconstruction from Sparse Representations: An Introduction to Compressed Sensing

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Digital Signal Acquisition

Reconstruction from Linear Measurements

Sparse Reconstructions

Fusing multiple representations

End

Extra Slides
Suppose we want to acquire some real world signal digitally. Applications abound:

- Digital cameras
- Medical imaging
- Sound recording
- Temperature measurements
- etc...

How many measurements do we need to take?
Classical answer: Nyquist-Shannon theorem. Sampling rate must be at least the Nyquist frequency $2W$.

This works for *bandlimited* signals. We can get by with fewer measurements if we know signals of interest are *sparse* or *compressible*; this is Compressive Sensing.
In CS we model the acquisition process by *linear measurements*. Given a measurement matrix $\Phi \in \mathbb{R}^{m \times n}$, form measurement

$$y = \Phi x.$$ 

This is the only information about $x$ that we can access.
The Basic Problem

How do we reconstruct a vector $x$ given only the measurement $y = \Phi x$?

Assume $\Phi \in \mathbb{R}^{m \times n}$ is full rank. Then either:

- $m \geq n$: system is *fully determined*

or

- $m < n$: system is *underdetermined*.
Overdetermined Case

If $\Phi \in \mathbb{R}^{m \times n}$ is full rank with $m \geq n$, unique solution to

$$y = \Phi x$$

is

$$x = (\Phi^*\Phi)^{-1}\Phi^* y.$$  

If the number of measurements is at least the length of $x$ then we can reconstruct $x$ exactly.
Underdetermined Case

More interesting: what if $\Phi \in \mathbb{R}^{m \times n}$ and $m < n$?

$$y = \Phi x$$

has infinitely many solutions. $\text{null}(\Phi)$ is nontrivial, and

$$\Phi x = \Phi(x + n), \forall n \in \text{null}(\Phi).$$

Can determine reconstruction only up to affine space

$$x + \text{null}(\Phi).$$
Least squares solutions

How do we pick the "best" solution in the affine space $x + \text{null}(\Phi)$? Simplest method: take solution with minimum $l^2$ norm (least squares)

$$x = \Phi^+ y = \Phi^* (\Phi \Phi^*)^{-1} y$$
Least squares solutions

Least squares is fast, reliable, but often gives poor reconstructions.

Figure: (left) original image, (right) least-squares reconstruction.
Better reconstructions

To improve on least squares, need more info about signals of interest. In CS we assume signals are sparse or compressible under some transform $\Psi$.

- Sparse: $\Psi x$ has only a few nonzero coefficients
- Compressible: well-approximated by sparse signal
We look for the *sparsest* solution in affine space $x + \text{null}(\Phi)$; solve

$$\arg\min_{x} ||\Psi x||_0$$

subject to

$$\Phi x = y$$
Theorem

If \( x \) is \( s \)-sparse and any \( 2s \) columns of \( A \) are linearly independent, then \( x \) is the unique \( s \)-sparse solution to \( y = Ax \).

Proof.

Suppose \( Ax = Az \). Then \( A(x - z) = 0 \), and \( x - z \) is \( 2s \)-sparse. Since any \( 2s \) columns of \( A \) are linearly independent, this means \( x - z = 0 \).
So, if any $2s$ columns $\Phi \Psi^{-1}$ are linearly independent, then there is a unique solution to the $l^0$ minimization problem for any $s$-sparse $x$. We can reconstruct sparse signals from far fewer than the full set of measurements this way!

**Example**

$\Phi \in \mathbb{R}^{4 \times 10}$ with columns taken from the $10 \times 10$ Fourier matrix satisfies the conditions of the theorem. Any 2-sparse signal in $\mathbb{R}^{10}$ can be recovered from just 4 frequency measurements.
Problems with $l^0$ minimization

Unfortunately, $l^0$ minimization is a non-convex optimization problem and is intractable in practice (NP hard in general). We need alternate methods to find sparse solutions.
- $l^2$ minimization: fast and reliable, but inaccurate for sparse signals
- $l^0$ minimization: optimal reconstructions for sparse signals, usually can’t be solved in practice.
Other Solution Methods

- $l^1$ minimization
- Orthogonal Matching Pursuit
- Iterative Hard Thresholding
Orthogonal Matching Pursuit

Orthogonal Matching Pursuit, or OMP: build sparse representation one basis vector at a time. Given $y = \Phi x$, look for an $s$-sparse solution $\alpha$

- Start with $\alpha = 0$.
- At each step, compute residual $r_k = y - \Phi \alpha$
- Choose column of $\Phi$ that is most correlated with $r_k$ and update $\alpha$
Iterative Hard Thresholding

Much faster than OMP; only works for very sparse signals.

\[ \alpha_{n+1} = H_s(\alpha_n + \psi \Phi^*(y - \Phi \psi^* \alpha_n)) \]

\(H_s\): hard thresholding operator, keep only s largest coefficients
If we can find multiple sparse solutions to $y = \Phi x$, we can often combine them into a better solution, e.g. by averaging.
RandOMP: Improving the OMP Solution

RandOMP randomizes OMP algorithm. At each step, update support of $\alpha$ randomly. Run RandOMP multiple times, take average of the sparse solutions we find.
Randomized IHTs

Can apply same idea as RandOMP to IHTs. Use randomized thresholding operator $\text{Rand}H_s$; choose large coefficients to keep from a distribution.
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Figure: Robert Crandall
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Randomized IHTs Reconstruction of a Very Noisy Image
Randomized IHTs Reconstruction of a Very Noisy Image

Figure: (left) IHTs, (right) RandIHTs Averaged Solutions
The End

Thanks to Dr. Ali Bilgin for advising this project.

Primary references:

- *An introduction to compressive sampling*, E. Candes and M. Wakin
- *A Plurality of Sparse Representations is Better than the Sparsest One Alone*, M. Elad and I. Yavneh
- *Robust uncertainty principles: Exact signal reconstruction from incomplete frequency information*, E. Candes, J Romberg, and T. Tao
- *Compressed sensing*, D. Donoho
- *Iterative Hard Thresholding for Compressed Sensing*, T. Blumensath and M. Davies
Coherence between orthonormal bases $\Phi, \Psi$ is

$$\mu(\Phi, \Psi) = \sqrt{n} \cdot \max(\phi_i, \psi_j)$$

Satisfies $\mu \in [1, \sqrt{n}]$. $\mu = 1$ is *maximal incoherence*. Want sparsity and measurement bases to be as incoherent as possible.
In compressed sensing we want the measurement/sparsity bases to be as incoherent as possible. Some examples of low-coherence pairs:

- Standard basis and Fourier basis; maximally incoherent
- Fourier and wavelet bases; largely incoherent
- Random bases are incoherent with any fixed basis with high probability
**Figure:** (left) original image, (right) sparse reconstruction