“*Esa era la* tricky part …”:
Students’ mathematical reasoning in a dual-language middle school classroom

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Introduction

Immigrant youth and the children of immigrants comprise the fastest growing segment of the current population in the United States (Suarez Orozco & Suarez-Orozco, 2001). However, many teachers continue to view English monolingualism as normative (Gunderson, 2004), and students’ varying degrees of bilingualism continue to be seen as hurdles to rather than resources for learning (Jimenez, 2001). In fact, bilingual or multilingual individuals typically participate in multiple-language communities, using one language for some functions and other languages for other functions and situations--in a complementary, rather than duplicative, fashion. At the same time, social fluency in a second language does not necessarily indicate facility with academic discourses required to succeed in school. Immigrant students who may have acquired English language proficiency to negotiate a myriad of social functions, e.g., helping their families negotiate complex household affairs successfully, may at the same time be at risk of failing academically (Orellana, Dorner, & Pulido, 2003; Rubinstein-Avila, 2004). As Ernst-Slavit, Moore and Maloney (2002) put concisely, “fluency in the hallway does not necessarily mean proficiency in the classroom” (p. 118).

Although English Language Learners (ELLs) are frequently lumped together into a single student group, several scholars have underscored the vast within-group heterogeneity among them (Bay-Williams & McGraw, in press; Gunderson, 2004; Jimenez, 2001; Moschkovich, 1999; Rubinstein-Avila, 2003; 2006; Valdés, 2001). Social construction of race and gender, age of arrival in the U.S., years of formal education, and rate of mobility (which are highly confounded with class) are among the many factors that shape students’ exposure to challenging mathematics curricula and associated academic outcomes. The strengths, needs, and circumstances of ELLs vary; some students have missed several years of formal schooling in their country of origin and enter middle school with only rudimentary arithmetic competencies, while others have already been exposed to algebra before coming to the U.S. (Moschovich, 1999).

Facility with academic language supports, and is developed through, the learning of particular subject matter. With respect to mathematics, scholars suggest that communication is a key element in the learning process (e.g., Campbell, Adams, & Davis, 2007). Also, due in part to current efforts to reform mathematics instruction, classrooms are expected to be increasingly communicative and language-rich (Adler, 1998). “Students are expected to communicate mathematically, both orally and in writing, and participate in mathematical practices, such as explaining solutions processes, describing conjectures, proving conclusion, and presenting arguments” (Moschkovich, 2002, p.190). The National Council of Teachers of Mathematics (NCTM) (2000) includes both communication and reasoning in its “process” standards for school mathematics. Specifically, students should make conjectures, develop mathematical arguments, communicate thinking to teachers and peers, and use language to express mathematical ideas (NCTM, 2000). In addition, the National Research Council (NRC) describes the need for students to develop adaptive reasoning abilities, i.e., “the capacity for logical thought, reflection, explanation, and justification” (NRC, 2001, p. 5). To be successful in the types of mathematical communication and reasoning describe above, students must be supported in building upon the language skills they bring to the classroom. A number of strategies are available to teachers including using concrete models, connecting unfamiliar words with familiar synonyms, alerting students to the existence of cognates, encouraging the use of visual imagery, and emphasizing written as well as oral language skills (Herman & Abedi, 2004, Rubinstein-Avila, 2006).

Mathematics teachers teaching in classrooms with a high number of ELLs may view their students’ language (e.g., reading, writing, speaking, and listening) knowledge as an impediment to learning, and as a result may move too quickly to symbolic representations, before students have had the opportunity to develop conceptual understanding. This of course limits ELLs’ deeper understanding of mathematics and ultimately their access to higher levels of mathematics (Garrison & Mora, 1999). At the same time, teachers may confuse the trajectories of students’ language development with the trajectories of their abilities to reason mathematically and assume that students will not be able to tackle challenging mathematics until they have become fluent in academic English.

In a recent critical essay, Orellana and Gutierrez (2006) suggested that although research is likely to start with a problem, such as the need to improve mathematics instruction for ELLs, researchers must reflect upon how they may be contributing to the view that ELLs *are* the problem. Instead of focusing only on how ELLs change within a given classroom context, Orellana and Gutierrez suggest that researchers focus on “how change occurs *both* in the participants and the contexts of their participation” (p.119; italics added). In this paper, we describe both the context and process of bilingual and ELL students’ participation in a mathematics classroom. Specifically, we report on the ways in which a dual-language middle school mathematics classroom provided immigrant Latino/a students the opportunity to develop and apply mathematical reasoning in both Spanish and English. Our research questions include:

1. How do students use language (English and Spanish) to discuss and solve problems, and to explain their reasoning to their peers and teachers?
2. What methods of reasoning do students employ?
3. How can mathematical tasks be constructed so as to support the development of students’ mathematical reasoning abilities?

At the time of this writing, the main thrust of our analysis has been on Research Question #2 as it applies to one small group of students. Research Questions #1 and #3 are touched upon in this paper, but will be more thoroughly explicated in future writings.

Theoretical framework

Our study is based on a view of learning as participation, and mathematical reasoning as an interactive, socially constructed, and situated activity. In the following paragraphs, we describe our orientation to learning and its implications for the study of reasoning, and also discuss the research base on mathematical reasoning and how it informs our work.

Following Mason and Spence (1999), we distinguish between knowing-to and knowing-about.

> Knowing-to is active knowledge which is present in the moment when it is required. To try to produce knowing-to, formal education focuses on forms of knowing which are

> easier to teach and to test: knowing-that (factual), knowing-how (techniques and skills), and knowing-why . . . Together these constitute knowing-about the subject. . . . The central problem of education is that knowing-about does not in itself guarantee knowing-to. (p. 135)

Thus, knowing-to is a conceptualization of knowledge as awareness and action. The actions students take as they participate in school mathematics might be to respond appropriately to questions posed by the teacher, decipher test questions and write answers using appropriate symbols, or share ideas and strategies with other students. In each case, the action is situated within and only has meaning as it relates to the contexts (social and cultural) in which it is embedded. Although we agree with Mason and Spence (1999) that the explicit focus of school is on "knowing-about," we view knowing school mathematics as successful participation in the process of doing school mathematics; in other words, knowing *is always* knowing-to. This knowing-to is visible as action, understood in its relationships to the actions of others, and taken as part of an evolving practice occurring within layers of context. This conceptualization of knowing is consistent with Cobb (1999) who questioned the idea of mathematical content existing separate from its use (in the classroom or elsewhere). He described mathematical content as emerging "as the collective practices of the classroom community evolved" (p. 31). Again, this is a view of mathematics as inherently and inevitably a socially and culturally situated activity.

From a "knowing-to" perspective, the ability to reason mathematically involves successfully coordinating multiple actions including (1) recognizing when a logical inference (or series of inferences) is called for, (2) recognizing the type and degree of justification needed, and (3) harnessing language, including mathematical terms and symbols, to develop and convey meanings. Reasoning can be investigated by observing the methods students use to solve mathematical problems and justify their solutions. Students' reasoning and justification methods provide evidence of the nature of their understanding of mathematical proof, including what it means to prove, how proving is accomplished, and what having proof implies.

Much is still not known about the development of reasoning and proof in mathematics classrooms. We seem to be at the stage of proposing models and categorization schemes, based on investigations of the thinking of relatively small numbers of students. In the following paragraphs, we provide an overview of (relatively) recent conceptualizations and findings that inform our research.

Reid (2002) proposed a model of reasoning that includes five attributes: need; kind of reasoning; formulation; formality; and target. This model captures multiple aspects of the reasoning process, including the reason for reasoning (need), the mode of reasoning (kind of reasoning, degree of mathematical formality), the audience (target), and the level of metacognition (formulation). Each of these attributes affects the nature of the dialogue through which the reasoning is accomplished, and thus the "moves" students utilize – e.g., questioning, responding, building or redirecting, and regulation (Burbules, 1993). Each of these moves can act to support or inhibit the development of mathematical reasoning among students, depending on their tone and purpose.

With respect to the kind of reasoning, Simon (1996) argued that the traditional conceptualization of student reasoning as inductive or deductive is incomplete. To these two categories, he added a third – transformational reasoning. Transformational reasoning is "the mental or physical enactment of an operation or set of operations on an object or set of objects that allows one to envision the transformations that these objects undergo and the set of results of these operations” (p. 201). It is reasoning that involves mentally visualizing actions. Transformational reasoning differs from inductive reasoning in that inductive reasoning requires the generation and accumulation of “outputs.” Conclusions are then drawn from comparing the nature of inputs and outputs and looking across outputs. Transformational reasoning involves examination of the system that creates the outputs and how the system “runs.”

The types of discursive practices likely to facilitate movement from informal reasoning to purposeful and appropriate use of inductive, deductive, and transformational reasoning are simply not known at this time. The role and function of proof in elementary and secondary mathematics classrooms continue to be the subject of debate as well. Important work has been done that with respect to students’ beliefs about proof (e.g., Harel & Sowder, 1998) and curricular and practice-related barriers to the development of proof (Dreyfus, 1999). Research on students’ understanding of proof indicates that students’ knowledge is often shaky, at best. Chazan (1993) found that some students believed that proof is just evidence and a correct, deductive proof may still be vulnerable to attack – for instance, by finding a counterexample. Other students believed that empirical evidence can prove something is true “for certain.” Still others believed the deductive proofs hold only for the specific diagrams to which they refer – these students did not see the proof as true for some more general class of mathematical objects. Balacheff (1999) raised an additional issue – whether students view counterexamples as merely objections or as refutations indicating a contradiction. Harel and Sowder’s (2007) review of the research on the teaching and learning of proof suggests that students may not have a strong understanding of basic principles of logic, tend to use specific examples to (attempt to) prove mathematical statements, and, overall, have little sense of the purposes or meanings of proof.

Harel and Sowder (1998; 2007) categorize students' beliefs about mathematical proof in terms of "proof schemes" which are context-dependent and consist "of whatever constitutes ascertaining and persuading for that person" (2007, p. 809). These researchers propose three hierarchical categories (listed here in ascending order): external conviction, empirical, and deductive. Within the external conviction category are authoritarian, ritual, and non-referential symbolic proof schemes. The bases of proof are the word of an authority (e.g., teacher, book), the "look" of the argument (e.g., two-column format), and manipulation of symbols without consideration of meaning, respectively. In each case, proof is not found within the argument itself, but rather from some external structure or source. Within the empirical category, proof is based on either perceptions (e.g., the appearance of a drawing), or induction based on one or more examples (e.g., the fact that a relationship holds true for several examples proves it will always hold true). The deductive category is divided into transformational and axiomatic proof schemes. "Transformational proof schemes share three essential characteristics: *generality, operational thought,* and *logical inference* (emphasis in original) (Harel & Sowder, 2007, p. 809). Generality refers to students' understanding that the intent is to create a proof that will hold in all cases without exception, operational thought refers to the ability to anticipate, and use as evidence, the outcomes of actions on mathematical objects, and logical inference refers to the knowledge that mathematical justifications rest upon rules of logic. Axiomatic proof schemes include both the characteristics of the transformational scheme and understandings of the role of axioms and differences among axiomatic systems.

Although the focus of our study is on student reasoning, Harel and Sowder's proof scheme framework is useful here because abilities and knowledge related to mathematical reasoning are inextricably linked to development of beliefs about, and abilities to generate, proofs. At the same time, relationships between reasoning and proof are complicated. For example, one cannot assume that the ability to reason deductively implies an understanding of the nature and role of deductive proof in mathematics. With respect to teacher practices, Harel and Sowder (2007) suggest that their hierarchy may be useful to teachers in planning instruction that would facilitate movement toward more sophisticated proof schemes; however, the nature of such instruction remains largely hypothetical at this time.

Mathematical reasoning always occurs in relation to some particular problem or task. In our study, the problem was that of representing the relationship between two variables – the row number associated with a set of stadium seats and the height above the ground of a person sitting in that row (see Appendix A). This is a *generational* task (Kieran,1996; 2007) in that it involves "the forming of expressions and equations that are the objects of algebra" (p. 713). Algebra is used as the language through which meanings generated within the problem context are expressed. Researchers who investigate the use of generational activities with beginning algebra students have found that students often have difficulty using algebra to generalize patterns (e.g., Warren, 2000). Mason (1996) described the possible influences of the instructional approach on students' generalizations, and Kieran (2007) suggests that further research is needed on the role of the teacher and the impact of classroom discussion on the learning of algebra.

It is important to note that, in algebra and elsewhere, mathematicians often mix deduction and experiment (Cuoco, Goldenberg, & Mark, 1996), and reasoning that eventually becomes part of a "proof" is developed through investigating mathematical objects and relationships. Proofs serve not only to convince and establish truth, but also to explain and connect ideas, and proving can lead to the discovery of new results. Research on student reasoning, such as the study described here, should take into consideration both the particular mathematical content of the reasoning (i.e., the generation of an algebraic representation of a particular problem situation) and the multiple roles reasoning and proving play in mathematics.

Methods

The findings we present in this paper are part of a larger study that examined mathematical reasoning as it occurs in middle school classrooms with high populations of ELLs. This larger study began in the fall of 2005 with observations of ten middle school mathematics teachers. These teachers were employed in two large school districts with high populations of Latino/a and ELL students. After conducting observations across all ten mathematics classrooms, we identified and continued to observe one focal teacher, Mrs. Lynch, during the spring of 2006. The selection of the focal teacher was based, in part, on the uniqueness of the teaching approach she utilized. Mrs. Lynch's class was part of a dual-language program in which the overall goal was to produce bilingual and biliterate learners. Such programs are neither new nor particularly rare in elementary schools, however few such programs extend throughout middle school. . In order for students who are classified as ELLs to participate in such a program in an English-only state, their parents/guardians are required to sign a waiver. This option is generally not widely publicized by school districts.

Embedded Contexts

Across educational research, classrooms are often mentioned only as a backdrop to the variables that account for achievement outcomes. Even within qualitative studies in mathematics education, the classroom context and the face-to-face interaction are often presented as decontextualized from other embedded contexts that shape them. Although an ethnographic study is likely to portray a particular classroom in more detail, it is our belief that no classroom is free from the social, political, and economic influences of the state, district and the school. Given the complex relationship between and among such embedded contexts, we provide the reader with layers of information that together create the context and help locate Mrs. Lynch's 7th/8th grade dual-language mathematics class and its bilingual students—during a time in which bilingual education across the U.S. is being eroded.

Thus, we begin by briefly describing the recent educational climate in the state of Arizona—especially as it relates to bilingual education and the education of ELLs. Then, we provide a brief description of the community and the school contexts, and lastly we describe the school's dual language program and Mrs. Lynch's multigrade mathematics class—our focal context.

State policy on the education of ELLs. Best described as a revision to California's Proposition 227, Proposition 203 was passed by Arizona voters in 2000. This proposition largely replaced bilingual programs with a one-year Structured English Immersion (SEI) program. Although the proposition was aimed at eliminating bilingual education, it did not outlaw it. Parents who wanted their children to continue to receive instructional support in Spanish had to apply for waivers that would exempt their children from the default SEI program (Combs et al., 2005).

Community and school contexts. This particular community, which we call *Saguaritos*, is just south of a mid-size southwestern city and has been historically populated by *Mexicano* origin families. While many new immigrant families settle in this community, some families have been living in the neighborhoods for several generations. There is also great variability in the years of formal schooling among students' parents; some are from rural Mexico, and may have only completed 2-3 years of formal schooling, others have completed high school and some even two-years degrees. Many parents work at the post office, airport, police department, in service industries, or are employed as staff members at the local schools. As is common in immigrant communities, some of the parents had been professionals in Mexico. Within the school's zip code, 10% of adults 25 years of age and older have college degrees, while the state average is 25%. The median income level for the school's zip code is $27,000 per year for a family of four. The student body of Secada Middle School numbers just under 1000: 86% Hispanic, 7% White, 3% Black, 1% Asian 1% and 1% Native American. Eighty percent of the students are eligible for free lunch and 8% for reduce lunch.

The dual language program at Secada Middle School. The bilingual program at Secada followed the "transitional" model (where the first language is slowly phased out as English competencies are being developed) until 2002/03 when it was redesigned into a dual-language program. During the time our study was conducted, the dual-language program employed 7 full-time, certified bilingual teachers (one of whom was also certified in Special Education), and the teachers worked collaboratively in planning their lessons. The program has had a stable enrollment of approximately 170 students (6th- though 8th-grade) every year since it started. Because of the shortage of teachers in the program, the 7th- and 8th-grade students are almost always combined across content areas.

Most of the students in the dual-language program are Spanish dominant and all are Latinos/as. About one-third of the students in the program were born in Mexico and have lived in the United States for less than three years. All of the core content classes are taught in English and Spanish—the language of instruction alternates daily. For language arts, students are grouped by level of English proficiency. All elective classes are taught in English. Approximately half of the 6th-graders who enroll at Secada come from the dual-language elementary feeder--*Cielito Lindo* [Beautiful Little Sky]. The dual-immersion program at *Cielito Lindo* replaced the transitional bilingual program in 2000.

Mrs. Lynch's mathematics class. In Mrs. Lynch's classroom students sit in clusters of 3 or 4; the classroom is spacious, with plenty of room to move around. The walls indicate that the classroom is a "a literacy-rich environment;" posters, photos, maps, mathematic symbols and equations, and words (in English and Spanish) crowd the walls, the two white boards, and even the door. Mathematics vocabulary words are held up with magnets on a side board, because according to Mrs. Lynch, "word walls are too static." Example vocabulary words such as rise, rate, linear equation, slope, vertical change, two step equation, variable, linear relationship, coefficient, and their equivalents in Spanish adorned the side boards on the week that the teaching experiment was conducted. The curriculum Mrs. Lynch follows is Connected Mathematics Project (CMP)—one of the NSF-funded middle school curricula. A key characteristic of this standards-based curriculum is its focus on the development of a deeper understanding of concepts through the bridging of concrete experiences with abstract representations. Following the dual-language tenants of the Secada program, Mrs. Lynch's mathematics class alternated between English and Spanish days. Although Mrs. Lynch encouraged students to express themselves in English during English days, Spanish was never banished. For example, if a student answered her query in Spanish, Mrs. Lynch would acknowledge it and immediately say, "OK. Gustavo, now how would you say that in English, please?" or she would acknowledge the student for the answer and then ask another student to translate the answer to English. Although each day designated the official language to be used for instruction and during whole-class time, students were always allowed to use either language when working within their small groups. Unlike language purists, who believe in strict adherence to one language or another, Mrs. Lynch encouraged students to think and develop mathematical discourse in whatever language they were most comfortable and felt most competent (regardless of the day).

Participants

The class in which the teaching experiment was conducted consisted of 16 female and 9 male students, and the majority of the students were classified as ELLs. According to the previous year's English proficiency test results, most students were assessed as advanced or intermediate. While students were bilingual, Spanish was their dominant language and clearly the language of choice among them. While a few were born in the United States, most were born in Mexico, and about one third of the students had been living in the U.S for 3 years or less. More than half of the students had attended the *Cielito Lindo* dual language program before coming to *Secada*. All four focal participants were born in Mexico, are Spanish dominant, and have lived in the U.S. for 3-5 years.

Mrs. Lynch is a European American teacher who has a Spanish bilingual certification and a middle school mathematics certification. She has 15 years of teaching experience, and has earned a Master's degree in education. Mrs. Lynch has received teaching awards for her work with ELL students and has been actively involved in teacher professional development.

Data Collection and Analysis

The findings presented here are based on the analysis of a small segment of the data collected as part of a teaching experiment conducted in Mrs. Lynch's classroom in the fall of 2006. During the experiment, one of the researchers engaged Mrs. Lynch's class in solving a non-routine task which required that students determine the relationship between two quantities – the bleacher number and the height above the ground for a given football stadium (see Appendix

A, also McGraw, Romero, & Krueger, 2006). The researcher/teacher was monolingual (English); thus, the activity was presented to the students in English and all whole-class activity, as well as all of the teacher's interactions with students, was conducted in English. The activities surrounding this task, e.g., comparing and analyzing various solutions, graphing the relationship, took place over three consecutive days. Data collected included videotape and audiotape of students working in small groups, copies of student work, and videotape of whole-class discussion. The focus of this paper is on a subset of the data collected – namely, the work of one group of four students during a portion of the first day of the activity.

The data pertinent to this paper consists of videotape and copies of student work of one small group of four students who were engaged in solving the stadium-seating problem (Appendix A). As previously stated, analysis of this data was guided by the attributes of reasoning suggested by Reid (2002), the concept of transformational reasoning (Simon, 1996), and the proof scheme hierarchy described by Harel and Sowder (2007). Brought to bear on this analysis were the diverse perspectives of the three authors – a mathematics educator, a mathematician, and a literacy specialist. Working as a team, we alternated between individual analysis of video, transcripts, and student work, and joint analysis focusing on variations in interpretations and the search for disconfirming evidence. Findings presented here are tentative, as they are based on the initial analysis of a relatively small segment of data.

Findings

In the interactions of one small group of students around the stadium-seating problem (Appendix A) we find evidence that middle-school students can engage in transformational reasoning and use examples and counterexamples to support their arguments. Critical to this process were two factors: (1) the students disagreed about what the solution should be and thus had a need for discussion, and (2) students continued to press one another for justification and explanation. In the following paragraphs, we describe how transformational reasoning was achieved through the group’s interactions, and as the problem unfolded in the classroom. Our description is organized chronologically, to preserve a sense of student reasoning as situated within both place and time.

The Interactive Constitution of Reasoning

The teacher set the stage by explaining the problem to the students (see Appendix A) and emphasizing that the goal was to find a method for finding the height of a person above the ground for any given bleacher height. The teacher did not explicitly suggest that students begin by trying specific values and making a table; however, she did not suggest that they should not do this. Her goal was to make sure students’ understood the goal of the activity, without suggesting a method for reaching that goal.

Christina, Marianna, Tanis, and Alberto began solving the problem by considering the relationship between the bleacher, or row, number and the total height. They proposed a series of operations for determining height, and responded to one another’s questions.

Alberto *Sumemos estos dos.*
［Let’s add these two. ］

Christina *Esos dos y luego cada multiplicación...luego sumamos por diez.*
[Those two and then each multiplication...then we add by tens. ］

Alberto *Aha. Entonces-*
[Aha. Then…]

Tanis *Si le vas sumando diez para que valla a cincuenta y así...*
[Yes, you keep adding ten so it goes to fifty and so on…]

Christina *-Cincuenta y siete mas diez y luego X lo que sea porque tienes que...oh*
diez mas diez y siete mas diezX (apunta esto en el papel)

[fifty-seven plus ten and then X or whatever because you have to…oh ten plus seventeen plus 10X (writes this down on the paper)]

Although the students are just beginning to work on the problem, they are already interacting with the problem at a general level. They immediately attempt to coordinate the changing height as evidenced in the diagram with the operations of addition and multiplication, and their focus is on the general relationship between movement along the steps and height above the ground. Although at this point the students are offering conclusions, we do note one warrant for the conclusions drawn in Tanis's use of the phrase "yes, you keep adding ten."

Next, variation among group members in the reading of the diagram caused disagreement about the solution. This variation seems to have been due to the fact that the computer projection of parts of the diagram, from which students were taking the measurements of 40, 17, and 10, were faint and therefore subject to misinterpretation. Marianna, Alberta, and Tanis seemed to believe that the height of the first bleacher was 40+17+10; thus, a correct representation of height (y) as a function of bleacher number (x) would be y=57+10x. Christina seemed to think that the height of the first bleacher was 40+17; thus, the addition of 10x would result in overcounting by ten.

Christina *No. mira porque si pones diez X vas a contar lo que son uno, dos, tres, cuatro, cinco, seis (contando los escalones en el papel) y luego y pero este ya no se cuenta (apuntando y refiriéndose al primer escalón)*
[No. look because if you put 10X you are going to count what is one, two, three, four, five, six (counting the steps on the paper) and then and but this one no longer counts (pointing and referring to the first step)]

Marianna *No pues no mas es...mira cincuenta y siete y...mas diez...mas diez...mas diez...mas diez.*
[No well its just…look fifty-seven and …plus ten…plus ten…plus ten…plus ten.]

Alberto *Cincuenta y siete...es este*
〔Fifty-seven…it's this one〕

Christina *Como va a hacer eso.*
[How can that be.]

Marianna *Pues por eso, por eso tenemos que multiplicarlo*
〔Well that's why, that's why we have to multiply it〕

Tanis *Pues por eso puede ser diez por diez...cincuenta y siete más cien para llegar al diez por ejemplo*
〔Well that is why it can be ten times ten …fifty-seven plus one hundred to get to ten for example〕

Christina *No mira porque si estas en el diez y si multiplicas por diez te van a dar diez más por este o menos diez puede ser puede ser menos diez porque este no cuenta mira*
〔No look because if you are at ten and you multiply by ten it is going to give you ten more for this or minus ten could be it could be minus ten because this doesn't count look〕

Tanis *¿Porque no cuenta? Si cuenta.*
[Why doesn't it count? it does count.]

Marianna *Claro que cuenta*
〔Of course it counts〕

Christina *Bueno, si cuenta pero ya lo estamos agregando aquí.*
〔Well good, it does count but we are now adding it here.]

In the face of a disagreement about the series of operations needed to obtain a correct solution, the students have continued to reason transformationally, justifying their conclusions by making general arguments based on the changes in height as represented by the diagram. In a certain sense they are "talking past" one another – each side believing that the other is seeing the diagram as they are. Interestingly, it is this accident – the low quality of the computer projection of the diagram – that has created an opportunity for us to observe the methods this group uses to justify their conclusions. When Christina's general argument doesn't prove convincing to the other students, she provided backing for her reasoning in the form of a specific example – "if you are at ten . . .". As she explained what would happen in the case of ten steps, she realized that "it could be minus ten" – not just for this example, but in general – as evidenced by the fact that she added a minus ten to 57+10*x* on her paper as she spoke.

At this point, Christina's is placed in the position of having to provide further backing for her conclusion that the first step "doesn't count."

Christina	*¿Porque mira si te dan el cinco aquí aquí estas, no? Vas a multiplicar cincuenta y siete más diez por cinco por cinco. (escribe esto en el papel) Entonces va a hacer cincuenta y siete mas cincuenta igual a ciento siete y no es ciento siete porque estas...no es ciento siete porque . . . entonces menos diez igual a noventa y siete...noventa y siete... ¿así tiene que ser que no?* [Why, look, if they give you the five you are here, here, right? You are going to multiply fifty-seven plus ten times five, times five (writes this down on the paper, second "times five" is said for emphasis) Then its going to be fifty-seven plus fifty equal to one hundred seven and its not one hundred seven because you are...its not one hundred seven because . . .then minus ten equal to ninety-seven…ninety-seven…this is the way it has to be right?]
Tanis	*¿Porque menos diez?* [Why minus ten?]
Christina	*Porque este no cuenta...porque este no cuenta como un diez (apunta al papel) este cuenta como diez y siete no diez...* [Because this one doesn't count…because this one doesn't count as a ten (points to paper) this one counts as seventeen and not ten]

Christina again tried to convince the other students that her solution was correct using a particular example, the fifth step. She attempted to create a counterexample, but could only argue that the answer of 107 was incorrect by reverting back to her general argument – that the first step counts as part of the 17, not as another 10. At this point, the teacher realized that the computer projection of the diagram was unclear, and clarified to students the lengths of 17 and 10 on the diagram and Christina was finally able to successfully make her argument.

Having established the need to subtract 10 and also an argument for the appropriateness of using the series of operations $57+10x-10$, the students moved on to what they viewed as the second task in the activity – calculating the heights of the 5th, 10th, and 50th rows (see Appendix A). The students seemed to view this as a test of their ability to use their algebraic expression to successfully complete a series of calculations. Ironically, the teacher's intent was that students use the specific cases of row 5 and 10 as a starting point for the investigation, if needed, and row 50 as a case for which a the generalization of "needing one less 10 than the number of rows" would be called for.

Once the students completed their calculations, Tanis made the following comment about the next statement of the worksheet, "Find a method that will allow you to determine how high you are above the ground for any row."

Tanis *Y la otra pues la maestra.*
[And the other well the teacher.]

Marianna *Y la otra pues….*
[And the other well…]

Tanis *La maestra que la haga (laughs)*
[The teacher can do it]

It seems that Tanis, at least momentarily, did not realize that the group had already accomplished this part of the activity. Perhaps because it came last, she assumed that it meant something different than what they had already done. At this point, the teacher came to the group and asked the students if they have found a solution. Compared to the level of reasoning and confidence displayed by the group previously, the group’s, and particularly Christina’s, interaction with the teacher was marked by more tentativeness.

Christina	I think we got it because we have to add...forty plus seventeen then multiply ten by each and then subtract ten because this one doesn't count.
Teacher	I think that makes sense
Christina	I think that's it.
Teacher	Good reasoning.
Tanis	Thank you.
Teacher	You’re welcome. (Teacher leaves the group)
Christina	*¿Entonces esta bien?* 〔It is right then?〕
Marianna	*Si*. [Yes]
Christina	*Esa era la* tricky part *de* subtracting *porque si no hace* subtract *te queda como si fuera un* step *mas.* 〔That was the tricky part of subtracting because if you don’t subtract it’s as if there was one more step.〕

The tentativeness that Christina expressed may have been due to the fact that the teacher was a relative stranger (an observer on several previous occasions and a teacher in the classroom for the first time that day). Her uncertainty about whether the group’s solution was correct may have been due to differences between the regular teacher’s methods of interacting with groups and the new teacher’s methods. The regular teacher, in our observations, often explicitly confirmed solutions, or suggested alternative one’s. Christina may also have been more confident in expressing her reasoning in her first language, Spanish, and may have simply expressed more tentativeness in speaking to the teacher because she was speaking in English. We imagine that differences in students' modes of expression and confidence across languages could be easily misinterpreted as lack of mathematical understanding or reasoning ability – this is an important but largely unexplored area in need of further research. Lastly, we also find it interesting that Christina code-switched for the first time during the activity after the teacher left the group. At the same time, she engaged in a meta-analysis of the problem – reflecting on the challenges it presented to the group.

In these episodes of group interaction, we find many examples of the students using transformational reasoning and working in ways that arguably foreshadow aspects of a transformational proof scheme – engaging with the problem in a general, or at least holistic, manner, connecting a series of operations with movements along the bleachers, and using logical inferences. However, we also found evidence at the end of the episode of students' desire for external verification, when Christina sought confirmation from the teacher of the correctness of the groups' solution. It is important to note that we do not attempt to make a case here that these students would exhibit transformational reasoning across problem situations. However, what we can say is that they are capable of engaging in such reasoning in situations of the type presented here.

Individual and Group Reasoning

As previously stated, analysis of the interactions among four students suggests that, as a group, they were capable of reasoning transformationally. Individually, we found that each student participated in ways that supported a high level of group reasoning. In the following paragraphs, we shift the analytical lens from the group to the individual-within-group, in order to consider how the actions of each of the four students contributed to the group's reasoning.

Christina. The most apparent examples of transformational reasoning were exhibited by Christina. Importantly, this reasoning emerged as part of the group's interaction process and because of the ways the students engaged together in questioning and pushing for understanding and consensus. Near the end of her group's discussion of the bleacher problem, Christina offered the following reason for why she felt her group's method was correct:

> I think we got it because we have to add . . . forty plus seventeen then multiply ten by each and then subtract ten because this one doesn't count.

In this case, Christina's ability to anticipate that there would be too many tens (if she added one 10 for each step) is evidence of her operational thought, and her ability to describe her reasoning without referring to a specific number of bleachers demonstrates the generality of her argument..

Earlier in her group's discussion, we found a more subtle example of Christina's reasoning related to the height of the x^{th} bleacher is $57+10x-10$.

> *El primero no es diez. El primero es diez y siete, ya el segundo es diez*. (Then a few seconds later.) *Si estas en eldiez, se cuentan nueve dieses.*
>
> [The first one is not ten. The first is seventeen, and the second is ten. . . . If you are at five, four tens are counted; if you are at ten, nine tens are counted.]

Although she cited particular examples, Christina emphasized that which is critical in order to reason generally – namely, the characteristic that distinguishes the first step. Her citation of specific bleachers was simply a way to clarify her point to her peers, as opposed to a way for her to understand the mathematics. If she had drawn the conclusion that the relationship between the number of bleachers and height was $57+10x-10$ based only on the statement, "if you are at five, four tens are counted; if you are at ten, nine tens are counted," her reasoning would have been significantly different. In such a (hypothetical) case, she would have made a general conclusion based upon two specific examples, thus demonstrating inductive reasoning characteristic of an empirical proof scheme.

Perhaps the most surprising episode in this activity was an exchange between Christina and the teacher near the end of the activity. After Christina explained her group's reasoning, the teacher praised the group, saying, "Good reasoning." Christina responded by asking:

> ¿Entonces esta bien?
>
> [It is right then?]

Despite both a logical explanation and an affirmation by the teacher of good reasoning, Christina wanted confirmation from the teacher that her group's method was correct. She ultimately sought explicit verification from an authority in order to be satisfied. She also conveyed uncertainty about the correctness of the solution by beginning her explanation with the phrase "I think we got it because . . . " Although this was only one incident, we posit that this example demonstrates the extent to which many students have learned to rely on external (authoritarian) sources to determine correctness of thinking. We are also aware, however, that Christina was speaking to the teacher in English and to her group in Spanish. Differences in tone and wording may have been due in part to speaking in one language versus the other.

Overall, the students in this group displayed a high degree of confidence in their own ability to find a solution to the problem, coupled with willingness to argue their positions based on operational thought and logical inference. Christina's descriptions of their thinking to the teacher, however, were quite tentative and uncertain by comparison. Although as previously stated, this tentativeness may have been due to reliance on external verification or to speaking in

English, an alternative hypothesis would be that Christina was fairly representing the fact that the group had considered other possible interpretations of the diagram, and other group members had initially disagreed with her interpretation. Other factors that may have influenced the tentativeness of her words include the fact that the teacher for the activity was a relative stranger (as discussed previously) and also the fact that the teacher did not explicitly confirm the correctness of the group's solution – which is something we had observed the regular teacher doing.

Marianna. Marianna played a critical role in the discussion in that she was willing to disagree with Christina and argue for her own way of thinking (which also seemed to be consistent with Alberto's and Tanis's ways of thinking). For example, Marianna said "No well it's just… look fifty-seven and… plus ten… plus ten… plus ten" and "that's why we have to multiply it." We see evidence here of her ability to connect the movement up the stadium seats with the operation of multiplication; thus we have some evidence (albeit a small amount) of her ability to reason transformationally.

Tanis. Tanis's role in the discussion was that of questioner and also provider of reasons. She repeatedly pressed others in the group to give reasons for their conclusions – she asked "But why do you say that?" "Why doesn't it count?" "Why minus ten?" and "but I don't understand." With respect to providing reasons, she argued, as Marianna did, that "you keep adding tens" and "that is why it can be ten times ten, fifty-seven plus one hundred to get to ten, for example." As with Marianna, we see a bit of evidence here suggesting that Tanis can reasoning transformationally (in this particular social and problem context) – she connected the addition of tens as one goes up the seats with multiplying and she understood that the calculation for row number 10 is one example of a more general idea.

Alberto. We have the least evidence about Alberto's thinking. We observed him actively listening throughout the discussion, and he contributed occasional ideas (e.g., "Let's add these two.), and confirmed the ideas of other's (e.g., "Fifty-seven, it's this one" and "Yes, ninety-seven").

In summary, we found that each student was an active participant in the social construction of the problem solution, and the constitution of transformational reasoning at the group level. By the end of their work, these students seemed to have a sense of "our group thinks this" rather than "I think this." The ways that they pressed one another for justification and continually tried to understand one another's ways of thinking are quite impressive; given what the research suggests about middle school students' mathematical knowledge and experience.

Discussion

In this activity, students' reasoning emerged as they worked to solve a specific task that was presented to the class in a specific way. The teacher emphasized verbally to students that their goal was to find a method for determining how high above the ground someone would be for any given bleacher. The students in the group we analyzed took up the task in the general manner in which it was given to them. They considered specific instances of the situation only as supports and justifications for their more general reasoning strategies. After finding and agreeing upon a solution ($57+10x-10$), the students then applied their method to the task of finding the height for the 5th, 10th, and 50th rows.

In the literature on tasks for developing student reasoning at the middle and secondary levels we find many examples of problems in which students are guided (explicitly or implicitly) through a process of reasoning that proceeds in the following manner: (1) students collect data for some relatively small number of examples and organize the data in a table, (2) student reason inductively from the data and make a conjecture about a more general relationship between quantities, (3) students verify that their conjecture works for numbers other than those in the table, and (4) students attempt to prove their conjecture will work in general (e.g., for all natural numbers, all rational numbers). Mason (1996) found that school problems involving generalization often emphasize table construction. “This approach in effect short-circuits all the richness of the process of generalization” (Kieran, 2007, p. 725). Moss (2005) goes a step further by suggesting that approaches that emphasize table construction could impede both recognition of general relationships and also the creation of algebraic representations.

The research on reasoning and proof suggests that students (and teachers) often exhibit empirical, rather than analytic, proof schemes (Harel & Sowder, 2007). It does not seem surprising to us that when relationships between quantities are determined through analysis of the relationships between numbers in a table, students will view the numbers in the table as “proof” that the relationship exists. Further, if students, for any reason, tend to exhibit empirical proof schemes, then the creation of tables as a way to develop conjectures seems counterintuitive as a method of supporting the development of deductive proof schemes.

The worksheet used by the teacher in this study (Appendix A) was based on the idea of generalizing based on specific examples. Under the diagram was written “How high above the ground would you be if you were sitting in the 5th row? The 10th row? The 50th row?” and then “Find a method that will allow you to determine how high you are above the ground for any row.” However, the teacher introduced the problem in a more general way and did not direct students’ attention to the instructions on the worksheet. This likely caused some students, such as those in the group reported on in this paper, to begin by considering the problem at a more general level. Given this, perhaps the ways in which the collection of data and the creation of tables are utilized in such problems bears re-consideration. One alternative approach would be to emphasize the use of specific instances as a way to “get a feel for” how variations in one quantity influence another, rather than emphasizing the creation of conjectures based on a small number of examples or based on the numbers in a table.

If mathematical reasoning is to be developed in middle and high school classrooms, then students must be put in situations in which a *need* for reasoning, and for communicating reasoning, exists. In our study, the need was created within the problem – the goal was given to students but the method of solving the problem was not. Students in the group we analyzed were able to move quickly toward a solution; however, this progress was slowed by a disagreement over the algebraic representation of the relationship between the row number and height above the ground. Although the basis for the disagreement was a faulty diagram, the fact that there was disagreement pushed reasoning to the center of the small-group interaction and raised the level of formulation, i.e., the level of metacognition (Reid, 2002). Importantly, students within the group were the target, or audience, for the reasoning. Within-group norms both supported and constrained students’ expressions of their reasoning. What counts as sufficient justification within such a group can be different from what students believe is appropriate outside the group – for example, when interacting with a teacher or speaking in front of the whole class. We see some evidence of such a difference in our data when comparing the conviction with which Christina spoke to her group members versus her hesitancy in speaking to the teacher.

In this study, students’ reasoning developed and became visible through the investigation of a mathematical relationship. Specific examples were used by students to support general claims and to clarify how operations should be used to represent the action of moving up the bleachers. Justification was used by the students to convince, but also to explain how and why.

Thus, the development of reasoning in this context was consistent with ways mathematicians use reasoning (Cuoco, Goldenberg, & Mark, 1996).

During the implementation of the activity, the teacher used several strategies to support language development, and build upon the language resources ELLs bring to the classroom. For example, the teacher began the lesson by projecting several images of a football stadium and bleachers and then displayed a pictorial representation of the bleachers without measurements. Students were given time to discuss with their peers the question that was being asked in a language of their choosing (i.e., Spanish). Following this, students shared, in English, the ideas and questions raised in their groups with the whole class. The need to speak in English during whole-group discussion was necessitated by the teacher's monolingualism – importantly, the students' ability to work across two languages was viewed as a resource and the use of only English during whole-class time as an accommodation. Although the focus of data analysis in this paper is on the verbal interaction of students in Spanish, students were provided with multiple opportunities to describe their reasoning both orally and in writing in both Spanish and English. In addition, prior to implementing the lesson, the researchers and Mrs. Lynch discussed mathematical terminology that would likely arise during the teaching experiment in order to determine how to best utilize synonyms and cognates throughout the activity.

Although the data analyzed for this paper was from one group and from one segment of a set of activities, it provides a useful example of what is possible with respect to mathematical communication and reasoning across two languages. Next steps in this research include analysis of data collected from other small groups, and a broader analysis of data collected across the three days of instruction.

References

Adler, J. (1998). A language of teaching dilemmas: Unlocking the complex multilingual secondary mathematics classroom. *For the Learning of Mathematics, 18*(1), 24-33.

Balacheff, N. (1999). Is argumentation an obstacle? Invitation to a debate. *The International Newsletter on Teaching and Learning of Mathematical Proo*f, May-June, 1-6.

Bay-Williams, J. & McGraw, R. (in press). Implementing a standards-based curriculum to improve student access to mathematics. In M. Meyer, C. Langrall, F. Arbaugh, D. Webb, & M. Hoover (Eds.), *A decade of middle school mathematics curriculum implementation: Lessons learned from the Show-Me Project*. Charlotte, NC: Information Age Publishing.

Burbules, N. C. (1993). *Dialogue in teaching: Theory and practice*. New York: Teachers College Press.

Campbell, A. E. Adams, V. M, & Davis, G. E. (2007). Cognitive demands and second language learners: A framework for analyzing mathematical instructional contexts. *Mathematical Thinking and Learning, 9*(1), 3-30.

Chazan, D. (1993). High school geometry students' justifications for their views of empirical evidence and mathematical proof. *Educational Studies in Mathematics, 24*, 359-387.

Cobb, P. (1999). Individual and collective mathematical development: The case of statistical data analysis. *Mathematical Thinking and Learning 1*(1), 5-43.

Combs, M. C. Evans, C., Fletcher, T., Parra, E. & Jimenez, A. (2005). Bilingualism for the children: Implementing a dual-language program in an English-only state. *Educational Policy, 19*(5), 701- 728.

Cuoco, A., Goldenberg, E. P., & Mark, J. (1996). Habits of mind: An organizing principle for mathematics curricula. *Journal of Mathematical Behavior*, *15*, 375-402.

Dreyfus, T. (1999). Why Johnny can't prove. *Educational Studies in Mathematics, 38,* 85-109.

Ernst-Slavit, G., Moore, M., & Maloney, C. (2002). Changing lives: Teaching English and literature to ESL students. *Journal of Adult & Adolescent Literacy, 48*(2), 118-128.

Garrison, L., & Mora, J.K. (1999). Adapting mathematics instruction for English language learners: The language-concept connection. In L. Ortiz-Franco, N. Hernández, & Y. De la Cruz (Eds.), *Changing the races of mathematics: Perspectives on Latinos* (pp. 35-48). Reston, VA: National Council of Teachers of Mathematics.

Gunderson, L. (2004). The language, literacy, achievement and social consequences of English-only programs for immigrant students. In C. M. Fairbanks, J. Worthy, B. Malloch, J. V. Hoffman, & D. L. Schallert (Eds.), *53rd Yearbook of the National Reading Conference*, pp. 1-27. Milwaukee, WI: National Reading Conference.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. *CBMS Issues in Mathematics Education, 7*, 234-283.

Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. In Frank K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning,* (pp. 805-842). Reston, VA: National Council of Teachers of Mathematics.

Herman, J. L., & Abedi, J. (2004). Issues in assessing English language learners' opportunities to learn mathematics. Center for the Study of Evaluation (CSE) Report 633. Los Angeles, CA: National Center for Research on Evaluation, Standards, and Student Testing.

Jiménez, R. T. (2001). “It’s a difference that changes us”: An alternative view of the language and literacy learning needs of Latino/a students. *The Reading Teacher, 54*(8), 736-742.

Kieran, C. (1996). The changing face of school algebra. In C. Alsina, J. Alvarez, B. Hodgson, C. Laborde, & A. Perez (Eds.), *Eighth International Congress on Mathematical Education: Selected lectures* (pp. 271-290). Seville, Spain: S.A.E.M. Thales.

Kieran, C. (2007). Learning and teaching algebra at the middle school through college levels. In Frank K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning*, (pp. 707-762). Reston, VA: National Council of Teachers of Mathematics.

Mason, J. (1996). Expressing generality and roots of algebra. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 65-86). Dordrecht, The Netherlands: Kluwer.

Mason, J., & Spence, M. (1999). Beyond mere knowledge of mathematics: The importance of knowing-to act in the moment. *Educational Studies in Mathematics, 38*, 135-161.

McGraw, R., Romero, D., & Krueger, R. (2006). How far up am I? The mathematics of stadium seating. *The Mathematics Teacher, 100*(4), 248-253.

Moschovich, J. N. (1999). Supporting the participation of English language learners in mathematics discussions. *For the Learning of Mathematics 19*(1)-11-19.

Moschovich, J. N. (2002). A situated and sociocultural perspective on bilingual mathematics learners. *Mathematical Thinking and Learning*, Special issue on Diversity , Equity and Mathematical learning, N. Nassir and P. Cobb (eds), 4(2 and 3), 189-212.

Moss, J. (2005, May). *Integrating numeric and geometric patterns: A developmental approach to young students’ learning of patterns and functions*. Paper presented at the annual meeting of the Canadian Mathematics Education Study Group, Ottawa.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Authors.

National Research Council. (2001). *Adding it up: Helping children learn mathematics.* J.Kilpatrick, J. Swafford, and B.Findell (Eds.). Mathematics Learning Study Committee, Center for Education, Division of Behavioral and Social Sciences and Education. Washington, DC: National Academy Press.

Orellana, M. F. & Gutierrez, K. (2006). AT LAST: What's the problem? Constructing different genres for the study of English Learners. *Research in the Teaching of English, 41*(1), 118-123.

Orellana, M. F., Dorner, L., & Pulido, L. (2003). Accessing Assets: Immigrant Youth as Family Interpreters. *Social Problems, 50*(5), 505-524.

Reid, D. (2002). Describing young children's deductive reasoning. *Proceedings of the 26th Annual Meeting of the International Group for the Psychology of Mathematics Education,* Norwich, England, July 21-26.

Rubinstein-Ávila, E. (2003). Facing reality: English language learners in middle school. *English Education, 35*(2), 122-136.

Rubinstein-Avila, E. (2004). Conversing with Miguel: An adolescent learner struggling with later literacy development. *Journal of Adolescent & Adult Literacy.*

Rubinstein-Avila, E. (2006). Connecting with Latino learners. *Educational Leadership.*

Simon, M. A. (1996). Beyond inductive and deductive reasoning: The search for a sense of knowing. *Educational Studies in Mathematics, 30*(2), 197-210.

Suárez-Orozco, C., Suárez-Orozco, M. (2001). *Children of immigration*: Cambridge, MA: Harvard University Press.

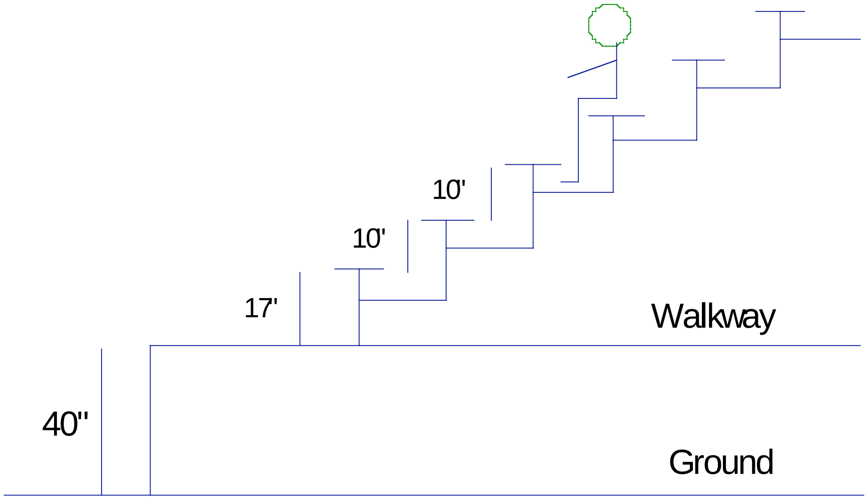
Valdés, G. (2001). *Learning and not learning English: Latino students in American schools*. New York, NY: Teachers College Press.

Warren, E. (2000). Visualisation and the development of early understanding in algebra. In T. Nakahara, & M. Koyama (Ed.), *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 273-280). Hiroshima, Japan.

Appendix A: Student handout for the stadium-seating problem

(NOTE: the lengths of 40, 17, and 10 were projected onto a screen at the front of the room, but were not included on the handout given to students)

How Far Up Am I? - Student Sheet 1 Name: ______________________



How high above the ground would you be if you were sitting in the 5th row? The 10th row? The 50th row?
Find a method that will allow you to determine how high you are above the ground for any row.