MATH 125:
LAST LECTURE

FALL 2009

1. Differential Equations

A **differential equation** is an equation involving an unknown function and it's derivatives. To **solve** a differential equation means to find a *function* that satisfies the given equation.

Question 1. *Consider the differential equation:*

$$4y - (y')^2 = -12y''\,.$$

This is an equation involving a function y and it's derivatives. Show that the function

$$y(x) = x^2 + 4x - 2$$

solves this differential equation.

Answer 1. *To check this, we calculate:*

$$y'(x) = 2x + 4$$

and

$$y''(x) = 2$$

Now check:

$$4y(x) - (y'(x))^2 = \big(4x^2 + 16x - 8\big) - 4(x+2)^2 = -24 = -12y''(x).$$

This function is a solution of this differential equation.

In general, there will be many solutions to a given differential equation. We often select a particular solution of interest by perscribing an initial value. This is often called **solving an initial value problem**.

Question 2. *Solve the following differential equation*

$$\frac{dy}{dx} = \sin(x) + 2\,.$$

Find the specific solution that satisfies:

$$y(3) = 5.$$

Answer 2. *To solve the above differential equation, we find an anti-derivative.*

$$y(x) = -\cos(x) + 2x + C\,.$$

Note the constant of integration.

To find the specific solution satisfying $y(3) = 5$*, we use this information to determine a specific constant of integration.*

$$5 = y(3) = -\cos(3) + 2 \cdot 3 + C \quad \Rightarrow \quad C = \cos(3) - 1\,.$$

In this case, the specific solution that satisfies $y(3) = 5$ *is given bu*

$$y(x) = -\cos(x) + 2x + \cos(3) - 1\,.$$

In general, this is how you solve an initial value problem.

Here is another example.

Question 3. *Show that*

$$y(x) = xe^{-x} + 2$$

is a solution of the initial value problem:

$$\frac{dy}{dx} = (1-x)e^{-x} \quad \textit{with } y(0) = 2.$$

Answer 3. *First, check the initial condition:*

$$y(0) = 0e^0 + 2 = 2.$$

This checks!

Now calculate:

$$y'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x}\,.$$

This checks too. So this is a solution of the above differential equation with this specific initial condition.

Question 4. *A water balloon launched from the roof of a building at $t = 0$ and has vertical velocity*

$$v(t) = -32t + 40 \quad \textit{in units of feet per second.}$$

Here $v > 0$ corresponds to upward motion.

a) If the roof of the building is 30ft above the ground, find an expression for the height of the water balloon.

b) What is the average velocity between $t = 1.5$ and $t = 3$ seconds?

c) A 6 foot tall person is standing on the ground. How fast is the water balloon falling when it strikes this person on the top of the head?

Answer 4. *a) We know the velocity:*

$$v(t) = -32t + 40\,.$$

Since $v(t) = s'(t)$, we need only find an anti-derivative. Clearly,

$$s(t) = -32\frac{t^2}{2} + 40t + C = -16t^2 + 40t + C.$$

We were told that the roof of the building is 30 feet above the ground. Since the balloon is launched from the roof, the position of the balloon at $t = 0$ is 30; i.e. $s(0) = 30$. This determines a specific constant of integration.

$$30 = s(0) = -16(0)^2 + 40(0) + C \quad \Rightarrow \quad C = 30.$$

The answer to question a) is then

$$s(t) = -16t^2 + 40t + 30.$$

b) Recall the definition of average velocity:

$$\textit{The Average velocity over [1.5,3]} = \frac{1}{3 - 1.5}\int_{1.5}^{3} v(t)dt$$

Clearly,

$$\int_{1.5}^{3}(-32t + 40)dt = \left(-16t^2 + 40t\right)\Big|_{t=1.5}^{t=3} = -48\,.$$

Thus

$$\textit{The Average velocity over [1.5,3]} = \frac{1}{3 - 1.5}\int_{1.5}^{3} v(t)dt = \frac{-48}{1.5} = -32 \textit{ ft/sec.}$$

c) First, we can easily determine the time t at which the balloon will hit the person on the head. This comes from solving $s(t) = 6$.

$$-16t^2 + 40t + 30 = 6 \quad \Rightarrow \quad 16t^2 - 40t - 24 = 0 \quad \Rightarrow \quad 16(t - 3)\left(t + \frac{1}{2}\right) = 0$$

The only positive solution is $t = 3$. To find the velocity at this time we evaluate

$$v(3) = -32(3) + 40 = -56 ft/sec$$

This answers part c).

Question 5. *If a car goes from 0 to 80 miles per hour in 6 seconds with a constant acceleration, what is the acceleration?*

Answer 5. *Well, we know that the acceleration is constant. Set it to be k.*

$$a(t) = k.$$

Since $a(t) = v'(t)$, we can find the velocity using anti-derivatives.

$$v(t) = kt + C\,.$$

We can determine the specific constant of integration using the fact that the initial velocity if 0.

$$0 = v(0) = k(0) + C \quad \Rightarrow \quad C = 0.$$

Thus, $v(t) = kt$. We also know that $v(6) = 80$. So,

$$80 = v(6) = k(6) \quad \Rightarrow \quad k = \frac{80}{6} = 13.33 \ \textit{meters/second}\,.$$

2. The Second Part of the Fundamental Theorem of Calculus

There are actually two important parts of the fundamental theorem of calculus. We have seen the first part. Briefly, this says the following:

$$F(b) - F(a) = \int_a^b F'(t)dt\,.$$

In words, one can use anti-derivatives to evaluate definite integrals.

Here is an example:

Question 1. *Evaluate the following definite integral.*

$$\int_0^\pi \sin(t)dt = ?$$

Answer 1. *Using the fact that the anti-derivative of* $\sin(t)$ *is* $-\cos(t)$*, the answer is easy.*

$$\int_0^\pi \sin(t)dt = (-cos(t))\,|_{t=0}^{t=\pi} = 1 + 1 = 2.$$

Notice that in the above definite integrals, the variable t appears. This variable is called *the variable of integration.* Of course, the results of these definite integrals **do not** depend on t (the variable of integration).

Once we have an anti-derivative, we can answer many questions.

$$\int_0^{\pi/4} \sin(t)dt = (-cos(t))\,|_{t=0}^{t=\pi/4} = -\frac{1}{\sqrt{2}} + 1 = \frac{\sqrt{2}-1}{\sqrt{2}}.$$

or

$$\int_0^{\pi/2} \sin(t)dt = (-cos(t))\,|_{t=0}^{t=\pi/2} = -0 + 1 = 1.$$

or

$$\int_0^{3\pi/2} \sin(t)dt = (-cos(t))\,|_{t=0}^{t=3\pi/2} = -0 + 1 = 1.$$

In fact, for any $x \geq 0$ we can evaluate

$$\int_0^x \sin(t)dt\,.$$

Of course, the value of this definite integral depends on the x value we choose. Hence, this quantity is a function of x. We can denote this function by f, i.e., set

$$f(x) = \int_0^x \sin(t)dt\,.$$

The second part of the Fundamental Theorem of Calculus discusses functions of this form.

Fundamental Theorem of Calculus (2nd Part) If f is a continuous function on an interval and a is any number in that interval, then the function F defined by

$$F(x) = \int_a^x f(t)dt \quad \text{for any x in the interval}$$

is an anti-derivative of f. In this case,

$$F'(x) = f(x).$$

Here is an example.

Question 2. *Consider the function*

$$Si(x) = \int_0^x \frac{\sin(t)}{t}\, dt\,.$$

Observe that Si is a function of x. It is not a function of t, the variable of integration!

Using the Fundamental Theorem of Calculus, calculate the derivative of:
a) $Si(x)$
b) $f(x) = xSi(x)$
c) $g(x) = Si(x^2)$.

Answer 2. *a) To calculate the derivative here, we just use the Fundamental Theorem (part 2).*

$$\frac{d}{dx}Si(x) = \frac{\sin(x)}{x}\,.$$

b) Here we use the product rule:

$$\frac{d}{dx}f(x) = Si(x) + x\frac{d}{dx}Si(x) = Si(x) + x \cdot \frac{\sin(x)}{x} = Si(x) + \sin(x)\,.$$

c) Here we use the chain rule. This is $Si(x)$ composed with x^2.

$$\frac{d}{dx}g(x) = \frac{d}{dx}Si(x)|_{x=x^2} \cdot \frac{d}{dx}x^2 = \frac{\sin(x^2)}{x^2} \cdot 2x = \frac{2\sin(x^2)}{x}\,.$$

Question 3. *Calculate*

$$\frac{d}{dx}\int_x^1 \ln(t)\, dt\,.$$

Answer 3. *First, note that this is an "upside down" integral (as far as the Fundamental Theorem is concerned). To calculate it's derivative, re-write it.*

$$\frac{d}{dx}\int_x^1 \ln(t)\, dt = -\frac{d}{dx}\int_1^x \ln(t)\, dt = -\ln(x) = \ln\left(\frac{1}{x}\right)\,.$$

Question 4. *Calculate*

$$\frac{d}{dt}\int_1^{\sin(t)} \cos(x^2)\,dx\,.$$

Answer 4. *The easiest way to differentiate this function is to recognize that it is a composition.*

Set

$$f(t) = \int_1^t \cos(x^2)\,dx \quad \Rightarrow \quad f'(t) = \cos(t^2)\,.$$

Set

$$g(t) = \sin(t) \quad \Rightarrow \quad g'(t) = \cos(t)\,.$$

Observe that

$$\int_1^{\sin(t)} \cos(x^2)\,dx = f(g(t))$$

and so

$$\frac{d}{dt}\int_1^{\sin(t)} \cos(x^2)\,dx = \frac{d}{dt}f(g(t)) = f'(g(t))\cdot g'(t) = \cos\left((\sin(t))^2\right)\cdot\cos(t)\,.$$

Here is a similar question.

Question 5. *Calculate*

$$\frac{d}{dx}\int_{\cos(x)}^3 e^{t^2}\,dt\,.$$

Answer 5. *Set*

$$f(x) = \int_x^3 e^{t^2}\,dt \quad \Rightarrow \quad f'(x) = \frac{d}{dx}\int_x^3 e^{t^2}\,dt = -\frac{d}{dx}\int_3^x e^{t^2}\,dt = -e^{x^2}\,.$$

Set

$$g(x) = \cos(x) \quad \Rightarrow \quad g'(x) = -\sin(x)\,.$$

Observe that

$$\int_{\cos(x)}^3 e^{t^2}\,dt = f(g(x))$$

and so

$$\frac{d}{dx}\int_{\cos(x)}^3 e^{t^2}\,dt = \frac{d}{dx}f(g(x)) = f'(g(x))\cdot g'(x) = e^{\cos^2(x)}\sin(x)\,.$$

3. Integration via Substitution

This is one of the most useful techniques of integration.

The basic idea is as follows:

(1) Take an ugly integral.
(2) Re-write the ugly integral as something you know how to integrate.
(3) Calculate that integral.
(4) Re-write your answer in terms of the original quantities.

Here is an easy problem:

Question 1. *Find the anti-derivative of*

$$\int x^3 dx = ?$$

We know the answer . . .

$$\int x^3 dx = \frac{x^4}{4} + C$$

Let's get this answer a different way.

Answer 1. *Let's re-write the integral as*

$$\int x^3\, dx = \int x^2 \cdot x\, dx$$

Introduce a new variable $u = x^2$. *Clearly,*

$$\frac{du}{dx} = 2x\,.$$

Let's write this symbolically as

$$du = 2x\, dx\,.$$

Now substitute in this new variable:

$$\int x^2 \cdot x\, dx = \int u \cdot \frac{1}{2} du = \frac{1}{2} \int u\, du\,.$$

Now, in some sense, the integral on the right-hand-side is easier to calculate. We get

$$\frac{1}{2} \int u\, du = \frac{1}{4} u^2 + C\,.$$

In terms of x, *what we have found is:*

$$\int x^3\, dx = \int x^2 \cdot x\, dx = \frac{1}{2} \int u\, du = \frac{1}{4} u^2 + C = \frac{1}{4} x^4 + C$$

when we substitute back the fact that $u = x^2$ *into our answer. This is integration by substitution.*

Here is a harder problem.

Question 2. *Calculate*

$$\int 3x^2 \cos(x^3)\, dx$$

Answer 2. *Try a substitution. Let*

$$u = x^3 \quad \Rightarrow \quad \frac{du}{dx} = 3x^2 \quad \Rightarrow \quad du = 3x^2 dx\,.$$

By re-grouping, it is easy to see that

$$\int 3x^2 \cos(x^3)\, dx = \int \cos(x^3) \cdot 3x^2 dx = \int \cos(u)\, du = \sin(u) + C = \sin(x^3) + C\,,$$

and we are done! As always, we can check our answers:

$$\frac{d}{dx} \sin(x^3) = \cos(x^3) \frac{d}{dx} x^3 = \cos(x^3) 3x^2\,,$$

as we wanted.

Technically, **substitution** is "reading the chain-rule backwards".

Recall: If f and g are differentiable functions, then

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)\,.$$

Thus if we integrate:

$$\int f'(g(x)) g'(x)\, dx = \int \frac{d}{dx} f(g(x))\, dx = f(g(x)) + C\,,$$

by the first part of the Fundamental Theorem of Calculus.

This looks too complicated, so we make a substitution. If we want to calculate:

$$\int f'(g(x)) g'(x)\, dx\,,$$

make a substitution: Let

$$u = g(x) \quad \Rightarrow \quad du = g'(x)\, dx$$

and then

$$\int f'(g(x)) g'(x)\, dx = \int f'(u)\, du = f(u) + C = f(g(x)) + C$$

as before.

The only hard part about substitution is recognizing what to substitute!

As above, often we substitute the "inside-function".

Question 3. *Calculate*

$$\int te^{t^2+1}\,dt$$

Answer 3. *Try a substitution. Let*

$$u = t^2 + 1 \quad \Rightarrow \quad du = 2t\,dt \quad \Rightarrow \quad \frac{1}{2}du = t\,dt\,.$$

By re-grouping, it is easy to see that

$$\int te^{t^2+1}\,dt = \int e^{t^2+1}\,t\,dt = \int e^u \frac{1}{2}du = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + C = \frac{1}{2}e^{t^2+1} + C\,,$$

and we are done! As always, we can check our answers:

$$\frac{d}{dt}\frac{1}{2}e^{t^2+1} = \frac{1}{2}e^{t^2+1}\frac{d}{dt}(t^2+1) = te^{t^2+1}\,,$$

as we wanted.

Question 4. *Calculate*

$$\int x^3\sqrt{x^4+5}\,dx$$

Answer 4. *Try a substitution. Let*

$$u = x^4 + 5 \quad \Rightarrow \quad du = 4x^3\,dx \quad \Rightarrow \quad \frac{1}{4}du = x^3\,dx\,.$$

By re-grouping, it is easy to see that

$$\int x^3\sqrt{x^4+5}\,dx = \int \sqrt{x^4+5}x^3\,dx = \int \sqrt{u}\,\frac{1}{4}du$$

and so

$$\frac{1}{4}\int \sqrt{u}\,du = \frac{1}{4}\cdot\frac{u^{3/2}}{3/2} + C = \frac{1}{6}(x^4+5)^{3/2} + C$$

and we are done! As always, we can check our answers:

$$\frac{d}{dx}\frac{1}{6}(x^4+5)^{3/2} = \frac{1}{6}\cdot\frac{3}{2}(x^4+5)^{1/2}\cdot 4x^3 = x^3\sqrt{x^4+5}\,,$$

as we wanted.

Question 5. *Calculate*

$$\int e^{\cos(\theta)}\,\sin(\theta)\,d\theta$$

Answer 5. *Try a substitution. Let*

$$u = \cos(\theta) \quad \Rightarrow \quad du = -\sin(\theta)\,d\theta \quad \Rightarrow \quad -du = \sin(\theta)\,d\theta\,.$$

By substituting, it is easy to see that

$$\int e^{\cos(\theta)}\,\sin(\theta)\,d\theta = -\int e^u\,du = -e^u + C = -e^{\cos(\theta)} + C,$$

and we are done! As always, we can check our answers:

$$\frac{d}{d\theta} - e^{\cos(\theta)} = -e^{\cos(\theta)}(-\sin(\theta)) = e^{\cos(\theta)}\sin(\theta)\,,$$

as we wanted.

Question 6. *Calculate*

$$\int \frac{e^t}{1+e^t}\,dt$$

Answer 6. *Try a substitution. Let*

$$u = 1 + e^t \quad \Rightarrow \quad du = e^t\,dt\,.$$

By substituting, it is easy to see that

$$\int \frac{e^t}{1+e^t}\,dt = \int \frac{1}{u}\,du = \ln(|u|) + C = \ln(1+e^t) + C\,,$$

and we are done! As always, we can check our answers:

$$\frac{d}{dt}\ln(1+e^t) = \frac{1}{1+e^t}e^t\,,$$

as we wanted.

Question 7. *Calculate*

$$\int \tan(\theta)\,d\theta$$

Answer 7. *Try a substitution. Let*

$$u = \cos(\theta) \quad \Rightarrow \quad du = -\sin(\theta)\,d\theta \quad \Rightarrow \quad -du = \sin(\theta)\,d\theta\,.$$

By re-writing, it is easy to see that

$$\int \tan(\theta)\,d\theta = \int \frac{\sin(\theta)}{\cos(\theta)}\,d\theta = -\int \frac{1}{u}du = -\ln(|u|) + C = -\ln(|\cos(\theta)|) + C,$$

and we are done!

You can also make substitutions into definite integrals:

Question 8. *Evaluate*

$$\int_0^2 x\, e^{x^2}\, dx$$

Answer 8. *One way to answer this problem is to use the Fundamental Theorem of Calculus. In this case, we first find an anti-derivative.*

Try a substitution. Let

$$u = x^2 \quad \Rightarrow \quad du = 2x\, dx \quad \Rightarrow \quad \frac{1}{2} du = x\, dx\,.$$

In this case, we find that

$$\int x\, e^{x^2}\, dx = \int e^{x^2}\, x\, dx = \int e^u \frac{1}{2}\, du = \frac{1}{2}\int e^u\, du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C$$

This shows that $\frac{1}{2}e^{x^2}$ *is an anti-derivative, and so*

$$\int_0^2 x\, e^{x^2}\, dx = \left(\frac{1}{2}e^{x^2}\right)\Big|_{t=0}^{t=2} = \frac{1}{2}(e^4 - 1)\,.$$

Question 9. *Evaluate*

$$\int_0^{\pi/4} \frac{\tan^3(\theta)}{\cos^2(\theta)}\, d\theta$$

Answer 9. *Try a substitution. Let*

$$u = \tan(\theta) \quad \Rightarrow \quad du = \sec^2(\theta)\, d\theta \quad \Rightarrow \quad du = \frac{1}{\cos^2(\theta)}\, d\theta\,.$$

In this case, we find that

$$\int \frac{\tan^3(\theta)}{\cos^2(\theta)}\, d\theta = \int u^3\, du = \frac{1}{4}u^4 + C = \frac{1}{4}\tan^4(\theta) + C$$

and so

$$\int_0^{\pi/4} \frac{\tan^3(\theta)}{\cos^2(\theta)}\, d\theta = \left(\frac{1}{4}\tan^4(\theta)\right)\Big|_{\theta=0}^{\theta=\pi/4} = \frac{1}{4}.$$

Question 10. *Evaluate*

$$\int_1^3 \frac{1}{5-x}\, dx$$

Answer 10. *Try a substitution. Let*

$$u = 5 - x \quad \Rightarrow \quad du = -\, dx \quad \Rightarrow \quad -du = dx\,.$$

In this case, we find that

$$\int \frac{1}{5-x}\, dx = -\int \frac{1}{u}\, du = -\ln(|u|) + C = -\ln(|5-x|) + C$$

and so

$$\int_1^3 \frac{1}{5-x}\, dx = (-\ln(|5-x|))\,\Big|_{x=1}^{x=3} = -\ln(2) + \ln(4) = \ln(2)\,.$$