An Example of Gaussian Elimination: Chapter 2

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An Example

Last class we considered the following example.

Ex 1: Consider the vectors

$$a = (-3, 2, 1, 4)\,, \quad b = (4, 1, 0, 2)\,, \quad \text{and} \quad c = (-10, 3, 2, 6)$$

all in $\mathbb{R}_4$. These vectors can be expanded in terms of the standard basis:

$$\begin{aligned} a &= -3e_1 + 2e_2 + e_3 + 4e_4 \\ b &= 4e_1 + e_2 + 0e_3 + 2e_4 \\ c &= -10e_1 + 3e_2 + 2e_3 + 6e_4 \end{aligned}$$

Example (cont.)

As we saw last class, the coefficient matrix corresponding to the vectors $\{a, b, c\}$ expanded in terms of the basis $\{e_1, e_2, e_3, e_4\}$ of $\mathbb{R}_4$ is:

$$A = \begin{pmatrix} -3 & 2 & 1 & 4 \\ 4 & 1 & 0 & 2 \\ -10 & 3 & 2 & 6 \end{pmatrix}$$

The result we proved at the beginning of class today shows that, in performing elementary row operations to A, we do not change $S(a, b, c)$. *Gaussian Elimination* is a method of choosing elementary row operations to perform on A that results in a simplified, yet row equivalent matrix A'. The corresponding system (for A') will also be easier to solve.

Gaussian Elimination

There are many ways to perform Gaussian Elimination. Here is the standard method.

1. Start with a matrix A.
2. Consider the first column of A.
3. If all entries in this column are zero, go to the next column (left to right).
4. Find the first non-zero entry in this column (top-down) and replace the first row of A with the row corresponding to this non-zero entry.
5. Use a type II elementary row operation, to eliminate (make zero) all entries in this column below this non-zero entry.
6. Proceed (left to right) to the next column and repeat the above exercise, looking for the first non-zero entry starting one row below the previous non-zero entry.

An Example

Consider the matrix

$$A = \begin{pmatrix} -3 & 2 & 1 & 4 \\ 4 & 1 & 0 & 2 \\ -10 & 3 & 2 & 6 \end{pmatrix}$$

In the first column, the first entry, -3, is non-zero. We will use it to eliminate the entries below it in the first column.

Step 1: The type II operation: $r_2 \mapsto r_2 + (4/3)r_1$ produces the following row equivalent matrix:

$$A \sim \begin{pmatrix} -3 & 2 & 1 & 4 \\ 0 & \frac{11}{3} & \frac{4}{3} & \frac{22}{3} \\ -10 & 3 & 2 & 6 \end{pmatrix}$$

An Example (cont.)

Step 2: The type II operation: $r_3 \mapsto r_3 + (-10/3)r_1$ produces:

$$\begin{pmatrix} -3 & 2 & 1 & 4 \\ 0 & \frac{11}{3} & \frac{4}{3} & \frac{22}{3} \\ -10 & 3 & 2 & 6 \end{pmatrix} \sim \begin{pmatrix} -3 & 2 & 1 & 4 \\ 0 & \frac{11}{3} & \frac{4}{3} & \frac{22}{3} \\ 0 & -\frac{11}{3} & -\frac{4}{3} & -\frac{22}{3} \end{pmatrix}$$

We now regard the first column as simplified.

Moving to the second column, we consider the entries in row 2 and below. The first entry is non-zero. We now use it to eliminate all the entries in the second column below this non-zero entry.

Step 3: The type II operation: $r_3 \mapsto r_3 + r_2$ produces:

$$\begin{pmatrix} -3 & 2 & 1 & 4 \\ 0 & \frac{11}{3} & \frac{4}{3} & \frac{22}{3} \\ 0 & -\frac{11}{3} & -\frac{4}{3} & -\frac{22}{3} \end{pmatrix} \sim \begin{pmatrix} -3 & 2 & 1 & 4 \\ 0 & \frac{11}{3} & \frac{4}{3} & \frac{22}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

An Example (summary)

The result we proved today demonstrates the following:

We started with the vectors

$$a = (-3, 2, 1, 4), \quad b = (4, 1, 0, 2), \quad \text{and} \quad c = (-10, 3, 2, 6)$$

all in $\mathbb{R}_4$.

We expanded these vectors in terms of the standard basis in $\mathbb{R}_4$ and obtained the coefficient matrix:

$$A = \begin{pmatrix} -3 & 2 & 1 & 4 \\ 4 & 1 & 0 & 2 \\ -10 & 3 & 2 & 6 \end{pmatrix}$$

An Example (summary)

We showed that the coefficient matrix is row equivalent to:

$$A \sim \begin{pmatrix} -3 & 2 & 1 & 4 \\ 0 & \frac{11}{3} & \frac{4}{3} & \frac{22}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Defining the vectors

$$a' = (-3, 2, 1, 4), \quad b' = \left(0, \frac{11}{3}, \frac{4}{3}, \frac{22}{3}\right), \quad \text{and} \quad c' = (0, 0, 0, 0)$$

we see that

$$S(a, b, c) = S(a', b', c') = S(a', b')$$

where the last equality is true because $c' = 0$.

It is easy to see that $\{a', b'\}$ is linearly independent. It is therefore a basis of $S(a, b, c)$.