Math 413 Homework 1 Key

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1. a. Prove that for each $k \geq 1$

$\star$ $$1+3+5+\cdots+(2k-1) = k^2$$

Base case:

$k=1 \Rightarrow 2k-1 = 1$

Left hand side $=1$ Right hand side $= (1)^2$ ✓

Induction:

Suppose $\star$ is true for $k \geq 1$

Consider

Note:
$2(k+1)-1-(2k-1)$
$= \cancel{2k}+2\cancel{-1}\cancel{-2k}\cancel{+1}$
$= 2$!

$$\underbrace{1+3+5+\cdots+(2k-1)}_{\substack{\uparrow \\ \text{induction}}} + (2(k+1)-1) = k^2 + 2k+1 \uparrow \text{simplify.}$$

Similarly $(k+1)^2 = k^2+2k+1$

This shows that $k+1$ is true.

1 b. Show that for each $n \geq 1$

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Base case $n = 1$

Left hand side $= 1$, Right hand side $\frac{1(2)(3)}{6} = 1$ ✓

Induction:

Suppose this is true for some $n \geq 1$.

Calculate

$$\underbrace{1^2 + 2^2 + \cdots + n^2}_{\text{By induction}} + (n+1)^2$$

$$= \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

$$= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6}$$

$$= \frac{(n+1)\left[n(2n+1) + 6(n+1)\right]}{6}$$

$$= \frac{(n+1)\left[2n^2 + n + 6n + 6\right]}{6}$$

$$= \frac{(n+1)(n+2)(2n+3)}{6}$$

$$= \frac{(n+1)(n+2)(2(n+1)+1)}{6}$$

This is the $n+1$ statement!

2. Let F be a field.
For $\alpha, \beta \in F$ with $\beta \neq 0$, define the symbol $\frac{\alpha}{\beta}$
as the solution of

$$\beta x = \alpha \quad \text{in } F.$$

Show that if $\beta \neq 0$ and $\delta \neq 0$, then

$$\frac{\alpha}{\beta} + \frac{\gamma}{\delta} = \frac{\alpha\delta + \beta\gamma}{\beta\delta}.$$

Pf:
Since $\beta \neq 0$ and $\delta \neq 0$, $\beta\delta \neq 0$.

(To prove this, suppose $\beta\delta = 0$. Since $\beta \neq 0$, β^{-1} exists.
Then $\beta\delta = 0 \Rightarrow \beta^{-1}\beta\delta = \beta^{-1}0 \Rightarrow 1\delta = 0 \Rightarrow \delta = 0$.
This is a contradiction.)

In this case, the right hand side is the solution of

$$\beta\delta x = \alpha\delta + \beta\gamma$$

Since $\beta \neq 0$, we can calculate:

$$\beta^{-1}\beta\delta x = \beta^{-1}(\alpha\delta + \beta\gamma)$$

$$\delta x = 1\delta x = \beta^{-1}\alpha\delta + \beta^{-1}\beta\gamma = \beta^{-1}\alpha\delta + 1\gamma$$
$$= \beta^{-1}\alpha\delta + \gamma$$

Thus

$$\delta x = \beta^{-1}\alpha\delta + \gamma$$

Since $\delta \neq 0$, we calculate that

$$\begin{aligned} x = 1x = \delta^{-1}\delta x &= \delta^{-1}(\beta^{-1}\alpha\delta + \gamma) \\ &= \delta^{-1}\beta^{-1}\alpha\delta + \delta^{-1}\gamma \\ &= \beta^{-1}\alpha\,\delta^{-1}\delta + \delta^{-1}\gamma \\ &= \beta^{-1}\alpha\cdot 1 + \delta^{-1}\gamma \\ &= \beta^{-1}\alpha + \delta^{-1}\gamma \\ &= \frac{\alpha}{\beta} + \frac{\gamma}{\delta} \end{aligned}$$

as claimed.

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3. For each integer $n \geq 1$ and each $0 \leq k \leq n$
The binomial coefficient is defined inductively by:

$n=1$: $\binom{1}{0} = 1$ and $\binom{1}{1} = 1$.

Suppose the coefficients for some $n-1 \geq 1$ are defined
i.e. $\binom{n-1}{k}$ for $0 \leq k \leq n-1$ are known

Set

$$\binom{n}{0} = \binom{n}{n} = 1 \quad \text{and}$$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \quad \text{for } k = 1, \dots, n-1.$$

Show that for any $\alpha, \beta \in F$

$$(\alpha+\beta)^n = \binom{n}{0}\alpha^n + \binom{n}{1}\alpha^{n-1}\beta + \cdots + \binom{n}{n}\beta^n \quad *$$

$$= \sum_{k=0}^{n} \binom{n}{k}\alpha^{n-k}\beta^k$$

pf by induction:

Base case: $n=1$

$$(\alpha+\beta)^1 = \alpha+\beta = \binom{1}{0}\alpha + \binom{1}{1}\beta \quad \checkmark$$

Suppose $*$ is true for some $n \geq 1$

Check

$$(\alpha+\beta)^{n+1} = (\alpha+\beta)(\alpha+\beta)^n$$

$$= (\alpha+\beta)\sum_{k=0}^{n}\binom{n}{k}\alpha^{n-k}\beta^k$$

$$= \sum_{k=0}^{n}\binom{n}{k}\alpha^{n-k+1}\beta^k + \sum_{k=0}^{n}\binom{n}{k}\alpha^{n-k}\beta^{k+1}$$

$$= \binom{n}{0}\alpha^{n+1} + \sum_{k=1}^{n}\binom{n}{k}\alpha^{n-k+1}\beta^k$$

$$+ \sum_{k=0}^{n-1}\binom{n}{k}\alpha^{n-k}\beta^{k+1} + \binom{n}{n}\beta^{n+1}$$

We want this to be equal to

$$\sum_{k=0}^{n+1}\binom{n+1}{k}\alpha^{n+1-k}\beta^{k} = \binom{n+1}{0}\alpha^{n+1} + \sum_{k=1}^{n}\binom{n+1}{k}\alpha^{n+1-k}\beta^{k} + \binom{n+1}{n+1}\beta^{n}$$

<u>Now</u> $\binom{n}{0}\alpha^{n+1} = \alpha^{n+1} = \binom{n+1}{0}\alpha^{n+1}$

and $\binom{n}{n}\beta^{n+1} = \beta^{n+1} = \binom{n+1}{n+1}\beta^{n}$

so the highest power of α and β agree in both expressions.

Moreover by definition

$$\sum_{k=1}^{n}\binom{n+1}{k}\alpha^{n+1-k}\beta^{k} = \sum_{k=1}^{n}\left[\binom{n}{k-1}+\binom{n}{k}\right]\alpha^{n+1-k}\beta^{k}$$

$$= \sum_{k=1}^{n}\binom{n}{k-1}\alpha^{n+1-k}\beta^{k} + \sum_{k=1}^{n}\binom{n}{k}\alpha^{n+1-k}\beta^{k}$$

Let $j=k-1$

$$= \sum_{j=0}^{n-1}\binom{n}{j}\alpha^{n-j}\beta^{k} + \sum_{k=1}^{n}\binom{n}{k}\alpha^{n+1-k}\beta^{k}$$

and so all terms agree!

4. Let F consist of $\{0,1\}$ with

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

Show that F is a field.

Clearly the sum and product are well-defined.

1) $\alpha+\beta=\beta+\alpha$ and $\alpha\beta=\beta\alpha$

$0+0=0$ ✓

$0+1=1 \quad 1+0=1$ ✓

$1+1=0$

$0\cdot0=0$

$0\cdot1=0 \quad 1\cdot0=0$

$1\cdot1=1$

2) $\alpha+(\beta+\gamma)=(\alpha+\beta)+\gamma \qquad (\alpha\beta)\gamma=\alpha(\beta\gamma)$

$\begin{cases} 0+(0+0)=0+0=0 \\ (0+0)+0=0+0=0 \end{cases}$ ✓

$\begin{cases} 1+(0+0)=1+0=1 \\ (1+0)+0=1+0=1 \end{cases}$

$\begin{cases} 0+(1+0)=0+1=1 \\ (0+1)+0=1+0=1 \end{cases}$

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$\begin{cases} 1+(1+1)=1+0=1 \\ (1+1)+1=0+1=1 \end{cases}$

$(0\cdot0)0=0\cdot0=0$

$0(00)=0\cdot0=0$

Any prod. containing zero is zero.

$(11)1=1\cdot1=1$

$1(11)=1\cdot1=1$ ✓

3. $\alpha(\beta+\gamma) = \alpha\beta + \alpha\gamma$

$$\begin{cases} 0(0+0) = 0(0) = 0 \\ 00 + 00 = 0+0 = 0 \end{cases} \qquad \begin{cases} 1(1+0) = 1(1) = 1 \\ 1\cdot1 + 1\cdot0 = 1+0 = 1 \end{cases}$$

$$\begin{cases} 1(1+1) = 1(0) = 0 \\ 1\cdot1 + 1\cdot1 = 1+1 = 0 \end{cases} \qquad \begin{cases} 1(0+1) = 1(1) = 1 \\ 1\cdot0 + 1\cdot1 = 0+1 = 1 \end{cases}$$

$$\begin{cases} 1(0+0) = 1\cdot0 = 0 \\ 1\cdot0 + 1\cdot0 = 0+0 = 0 \end{cases} \qquad \begin{cases} 0(1+0) = 0(1) = 0 \\ 0\cdot1 + 0\cdot0 = 0+0 = 0 \end{cases}$$

$$\begin{cases} 0(0+1) = 0(1) = 0 \\ 0\cdot0 + 0\cdot1 = 0+0 = 0 \end{cases}$$

4. $0 \in F$

$0+0=0$ and $1+0=1$

5. $-0 = 0$ and $-1 = 1$

6. $1 \neq 0$ $\quad 1\cdot1 = 1$ $\quad 1\cdot0 = 0$

7. $1 \neq 0$ $\quad 1\cdot1 = 1.$ $\quad (1)^{-1} = 1.$

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2. $a = (-1,2,1)$, $b = (2,1,-3)$, $c = (0,1,0)$

a) $x + a = b \Rightarrow x = b - a = (2,1,-3) - (-1,2,1)$
$= (3,-1,-4)$

b) $2x - 3b = c \Rightarrow 2x = c + 3b$

$x = (2)^{-1}(c+3b)$
$= (2)^{-1}((6,4,-9))$
$= (3, 2, -9/2)$

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2. c) $b + x = a - 2c$

$$x = a - 2c - b$$
$$= (-1, 2, 1) + (0, -2, 0) + (-2, -1, 3)$$
$$= (-3, -1, 4)$$

3. This we discussed in class

Please check!