Math 413 Hw 2

pgs 32-33 1, 3, 4, 5, 7, and 8

1. Consider the following subsets of $\mathbb{R}_n$. Are they subspaces?

a) $W = \{(\alpha_1, \dots, \alpha_n) \in \mathbb{R}_n \mid \alpha_1 = 1\}$

Let $u, v \in W$.
Then $u = (1, \alpha_2, \dots, \alpha_n)$
$v = (1, \alpha_2', \dots, \alpha_n')$
$u + v = (2, \alpha_2 + \alpha_2', \dots, \alpha_n + \alpha_n') \notin W$

No.

b. $W = \{(\alpha_1, \dots, \alpha_n) \in \mathbb{R}_n \mid \alpha_1 = 0\}$

Let $u, v \in W$. Then $u = (0, \alpha_2, \dots, \alpha_n)$
$v = (0, \beta_2, \dots, \beta_n)$
and
$u + v = (0, \alpha_2 + \beta_2, \dots, \alpha_n + \beta_n) \in W$.

Let $u \in W$ and $\alpha \in \mathbb{R}$. Then $u = (0, \alpha_2, \dots, \alpha_n)$
and
$\alpha u = (0, \alpha \cdot \alpha_2, \alpha \cdot \alpha_3, \dots, \alpha \cdot \alpha_n) \in W$

Yes.

c.

$$W = \{(\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{R}_n \mid \alpha_1 + 2\alpha_2 = 0\}.$$

Let $u, v \in W$. Then $u = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $\alpha_1 + 2\alpha_2 = 0$
and $v = (\beta_1, \beta_2, \dots, \beta_n)$ and $\beta_1 + 2\beta_2 = 0$

$\Rightarrow$ $u + v = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots, \alpha_n + \beta_n)$

and $(\alpha_1 + \beta_1) + 2(\alpha_2 + \beta_2) = (\alpha_1 + 2\alpha_2) + (\beta_1 + 2\beta_2)$
$= 0 + 0 = 0$

Let $u \in W$ and $\alpha \in \mathbb{R}$. Then $u = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $\alpha_1 + 2\alpha_2 = 0$

Clearly

$$\alpha u = (\alpha \cdot \alpha_1, \alpha \cdot \alpha_2, \dots, \alpha \cdot \alpha_n)$$

and

$$\alpha \cdot \alpha_1 + 2\alpha \cdot \alpha_2 = \alpha(\alpha_1 + 2\alpha_2) = \alpha \cdot 0 = 0.$$

yes.

d. $W = \{(\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{R}_n \mid \alpha_1 + \alpha_2 + \cdots + \alpha_n = 1\}.$

Clearly $u = (1, 0, \dots, 0) \in W$ and $v = (0, 1, 0, \dots, 0) \in W$

But $u + v = (1, 1, 0, \dots, 0) \notin W$.

No.

e. $W = \{(\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{R}^n \mid A_1\alpha_1 + A_2\alpha_2 + \dots + A_n\alpha_n = 0\}$

Let $u, v \in W$. $u = (\alpha_1, \alpha_2, \dots, \alpha_n)$ with $\sum_{j=1}^{n} A_j\alpha_j = 0$
$v = (\beta_1, \beta_2, \dots, \beta_n)$ with $\sum_{j=1}^{n} A_j\beta_j = 0$

Then $u + v = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots, \alpha_n + \beta_n)$

and $\sum_{j=1}^{n} A_j(\alpha_j + \beta_j) = \sum_{j=1}^{n} A_j\alpha_j + \sum_{j=1}^{n} A_j\beta_j = 0 + 0 = 0.$

Let $u \in W$ and $\alpha \in \mathbb{R}$. Then $u = (\alpha_1, \alpha_2, \dots, \alpha_n)$ with $\sum_{j=1}^{n} A_j\alpha_j = 0$

$\alpha u = (\alpha\alpha_1, \alpha\alpha_2, \dots, \alpha\alpha_n)$

and $\sum_{j=1}^{n} A_j \alpha \cdot \alpha_j = \alpha \sum_{j=1}^{n} A_j\alpha_j = \alpha \cdot 0 = 0.$

yes!

f. $W = \{(\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{R}^n \mid A_1\alpha_1 + A_2\alpha_2 + \dots A_n\alpha_n = B\}$

The case of $B = 0$ is above. If $B \neq 0$,
then let $u \in W$, i.e. $u = (\alpha_1, \alpha_2, \dots, \alpha_n)$
and $\sum_{j=1}^{n} A_j\alpha_j = B.$

Clearly $2u = (2\alpha_1, 2\alpha_2, \dots, 2\alpha_n)$
satisfies $\sum_{j=1}^{n} A_j 2\alpha_j = 2\sum_{j=1}^{n} A_j\alpha_j = 2B \neq B$ (Since $B \neq 0$)

No!

g. Let $W = \{(\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{R}^n \mid \alpha_1^2 = \alpha_2\}$.

Note $u = (1, 1, 0, \dots, 0) \in W$
$v = (-1, 1, 0, \dots, 0) \in W$

but $u + v = (0, 2, 0, \dots, 0) \notin W$. No!

3 Which of these subsets of $C(\mathbb{R})$ are subspaces

a. The set of all polynomials in $C(\mathbb{R})$. Denote this set by $P(\mathbb{R})$.

Let $f, g \in P(\mathbb{R})$.
Then $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$
and $g(x) = b_m x^m + b_{m-1} x^{m-1} + \dots + b_1 x + b_0$.

Take $N = \max[n, m]$. Then

$$f(x) + g(x) = c_N x^N + c_{N-1} x^{N-1} + \dots + c_1 x + c_0$$

where $c_j = \begin{cases} a_j & \text{if } N = n \text{ and } j > m \\ a_j + b_j & \text{for } 0 \le j \le m. \end{cases}$

or

$c_j = \begin{cases} b_j & \text{if } N = m \text{ and } j > n \\ a_j + b_j & \text{for } 0 \le j \le n \end{cases}$

Similarly if $f \in P(\mathbb{R})$ and $\alpha \in \mathbb{R}$, then

$$\alpha f(x) = \alpha a_n x^n + \alpha a_{n-1} x^{n-1} + \dots + \alpha a_1 x + \alpha a_0 \in P(\mathbb{R})$$ yes!

b. $W = \{f \in C(\mathbb{R}) \mid f(1/2) \in \mathbb{Q}\}$.

Not: $f(x) = x$ satisfies $f \in W$.
However $2\sqrt{2}f \notin W$ since
$2\sqrt{2}f(1/2) = \sqrt{2} \notin \mathbb{Q}$.

c. $W = \{f \in C(\mathbb{R}) \mid f(1/2) = 0\}$.

Let $f, g \in W$ then $(f+g)(1/2) = f(1/2) + g(1/2) = 0$
Let $f \in W$ and $\alpha \in \mathbb{R}$, then
$\alpha f(1/2) = \alpha \cdot 0 = 0$

yes.

d. $W = \{f \in C(\mathbb{R}) \mid \int_0^1 f(t)\,dt = 1\}$.

Not: $f(t) = 2t$ satisfies $f \in W$
However $2f \notin W$. No.

e. $W = \{f \in C(\mathbb{R}) \mid \int_0^1 f(t)\,dt = 0\}$.

Let $f, g \in W$. Then
$\int_0^1 (f+g)(t)\,dt = \int_0^1 f(t)\,dt + \int_0^1 g(t)\,dt = 0$

Let $f \in W$ and $\alpha \in \mathbb{R}$ Then
$\int_0^1 \alpha f(t)\,dt = \alpha \int_0^1 f(t)\,dt = 0$ yes.

f. $W = \{ f \in C(\mathbb{R}) \mid \frac{df}{dt} = 0 \}$

Let $f, g \in W$. Then $\frac{d}{dt}(f+g) = \frac{df}{dt} + \frac{dg}{dt} = 0 + 0 = 0$.
Let $f \in W$ and $\alpha \in \mathbb{R}$, then $\frac{d}{dt}(\alpha f) = \alpha \cdot \frac{d}{dt} f = 0$.
yes.

g. $W = \{ f \in C(\mathbb{R}) \mid \alpha f'' + \beta f' + \gamma f = 0$ for some $\alpha, \beta, \gamma \in \mathbb{R} \}$

Let $u, v \in W$. Then

$\alpha (u+v)'' + \beta (u+v)' + \gamma (u+v)$
$= (\alpha u'' + \beta u' + \gamma u) + (\alpha v'' + \beta v' + \gamma v)$
$= 0 + 0 = 0$ ✓

If $u \in W$ and $\lambda \in \mathbb{R}$, then   Yes

$\alpha (\lambda u)'' + \beta (\lambda u)' + \gamma (\lambda u)$
$= \lambda (\alpha u'' + \beta u' + \gamma u)$
$= \lambda 0 = 0$.

h. $W = \{ f \in C(\mathbb{R}) \mid \alpha f'' + \beta f' + \gamma f = g$ for $\alpha, \beta, \gamma \in \mathbb{R}$ and $g \in C(\mathbb{R}) \}$

If $g = 0$, this is example g. If $g \neq 0$, then
Let $u, v \in W$. Then
$\alpha (u+v)'' + \beta (u+v)' + \gamma (u+v)$
$= (\alpha u'' + \beta u' + \gamma u) + (\alpha v'' + \beta v' + \gamma v) = 2g \neq g$

No

since $g \neq 0$.

4. L.D. or L.I. ?

a. $(1,1)$ and $(2,1)$

Suppose $(1,1) = \lambda(2,1) \Rightarrow 1 = 2\lambda$ and $1 = \lambda$

Not possible

Suppose $(2,1) = \lambda(1,1) \Rightarrow 2 = \lambda$ and $1 = \lambda$

Not possible

$\Rightarrow$ L.I.

b. $(1,1)$, $(2,1)$, and $(1,2)$

Let $\alpha = -3$, $\beta = 1$, and $\gamma = 1$

$\alpha(1,1) + \beta(2,1) + \gamma(1,2)$

$= (-3,-3) + (2,1) + (1,2)$

$= (0,0)$ ✓ Since they are not all zero $\Rightarrow$ L.D.

c. $(0,1)$ and $(1,0)$

Let $\alpha, \beta \in \mathbb{R}$ and suppose

$\alpha(0,1) + \beta(1,0) = 0$

$\Rightarrow (0,\alpha) + (\beta,0) = 0$

$\Rightarrow (\beta,\alpha) = 0$

$\Rightarrow \beta = \alpha = 0 \Rightarrow$ L.I.

d. Fix $\alpha, \beta \in \mathbb{R}$. and consider $(0,1)$, $(1,0)$, and (α,β)

Take

$\beta(0,1) + \alpha(1,0) - (\alpha,\beta) = 0$

since $-1 \neq 0$, These vectors are L.D.

e. $(1,1,2), (3,1,2), (-1,0,0)$

Let $\alpha, \beta, \gamma \in \mathbb{R}$ and set

$$\alpha(1,1,2) + \beta(3,1,2) + \gamma(-1,0,0) = 0$$

$$\Rightarrow \begin{aligned} \alpha + 3\beta - \gamma &= 0 \\ \alpha + \beta &= 0 \\ 2\alpha + 2\beta &= 0 \end{aligned} \qquad \begin{aligned} \alpha &= 1 \\ \beta &= -1 \\ \gamma &= 2 \end{aligned}$$

L.D.

f. $(3,-1,1), (4,1,0), (-2,-2,-2)$

Let $\alpha, \beta, \gamma \in \mathbb{R}$ and set

$$\alpha(3,-1,1) + \beta(4,1,0) + \gamma(-2,-2,-2) = 0$$

$3\alpha + 4\beta - 2\gamma = 0 \Rightarrow 3(2\gamma) + 4(4\gamma) - 2\gamma = 0 \Rightarrow 20\gamma = 0$

$-\alpha + \beta - 2\gamma = 0 \Rightarrow \beta = 4\gamma$

$\alpha + 0 - 2\gamma = 0 \Rightarrow \alpha = 2\gamma$

$\Rightarrow \gamma = 0$
$\Rightarrow \alpha = 0$
$\Rightarrow \beta = 0$
$\Rightarrow$ L.I.

g. $(1,1,0), (0,1,1), (1,0,1), (1,1,1)$

Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and set

$$\alpha(1,1,0) + \beta(0,1,1) + \gamma(1,0,1) + \delta(1,1,1) = 0$$

$$\Rightarrow \begin{aligned} \alpha + 0 + \gamma + \delta &= 0 \\ \alpha + \beta + 0 + \delta &= 0 \\ 0 + \beta + \gamma + \delta &= 0 \end{aligned}$$

Take $\gamma + \delta = 1$
$\Rightarrow \alpha = -1, \beta = -1$
$\Rightarrow \delta = 2 \Rightarrow \gamma = -1$

These are L.D. $\alpha = -1, \beta = -1, \gamma = -1, \delta = 2$.

5. Find a set of L.I. generators for

$W = \{ (x_1, x_2, x_3) \in \mathbb{R}_3 \mid x_1 - x_2 + x_3 = 0 \}$.

Note: $a = (1,1,0)$ and $b = (0,1,1)$ satisfy $a, b \in W$.

Claim: $W = S(a,b)$

pf:

It is easy to check that $W \subset \mathbb{R}_3$ is a subspace.
Since $a, b \in W$, Theorem (4.5) $S(a,b) \subset W$.
We need only show that $W \subset S(a,b)$.
Let $x \in W$. Then $x = (x_1, x_2, x_3)$ and $x_1 - x_2 + x_3 = 0$.

- If $x = (0,0,0)$, then $x = 0a + 0b \in S(a,b)$ and we are done.

- If $x_1 \neq 0$, then calculate:

$$x - x_1 a = (x_1, x_2, x_3) - (x_1, x_1, 0)$$

$$= (0, x_2 - x_1, x_3)$$

since $x_1 - x_2 + x_3 = 0$

$$= (0, x_3, x_3)$$

$$= x_3 b$$

i.e. $x = x_1 a + x_3 b$ with $x_1 \neq 0 \Rightarrow x \in S(a,b)$.

• If $x_1 = 0$ but $x \neq 0$, then since $x_1 - x_2 + x_3 = 0$

i.e. $-x_2 + x_3 = 0$

or $x_3 = x_2$

we must have that $x_2 = x_3 \neq 0$.

In this case

$$x = (0, x_2, x_3) = (0, x_2, x_2) = x_2 b \quad \text{and } x_2 \neq 0$$

$\Rightarrow x \in S(a, b)$.

Thus $W = S(a, b)$!

Claim: $\{a, b\}$ is L.I.

Check: Suppose $\lambda \in \mathbb{R}$ satisfies

$a = \lambda b \Rightarrow (1, 1, 0) = \lambda(0, 1, 1)$

$\Rightarrow 1 = 0 \quad 1 = \lambda \quad 0 = \lambda$ ↯.

Suppose $\lambda \in \mathbb{R}$ satisfies

$b = \lambda a \Rightarrow (0, 1, 1) = \lambda(1, 1, 0)$

$\Rightarrow 0 = \lambda, \ 1 = \lambda, \ 1 = 0$ ↯

$\Rightarrow$ L.I.

7. Let V be a vector space over F.

Let $W \subset V$ be a subspace.

Let $U \subset V$ be a subspace.

Consider $X = W \cap U \subset V$.

Let $x, y \in X$. Then $x \in W$ and $x \in U$

and $y \in W$ and $y \in U$

$\Rightarrow x + y \in W$ (since W is a subspace) and $x + y \in U$ (since U is a subspace)

$\Rightarrow x + y \in X$.

Let $x \in X$ and $\alpha \in F$.

Then $x \in W$ and $x \in U$

$\Rightarrow$ $\alpha x \in W$ (since W is a subspace)

$\Rightarrow$ $\alpha x \in U$ (since U is a subspace)

$\Rightarrow$ $\alpha x \in X$

$\Rightarrow$ $X = W \cap U$ is a subspace of V.

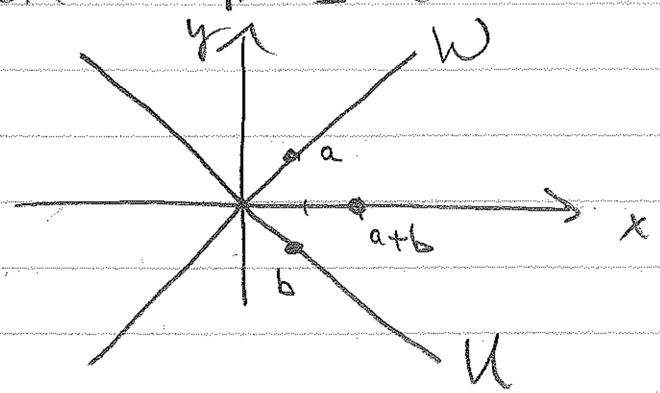
8. No. Consider $V = \mathbb{R}_2$ over $\mathbb{R}$.

Let $W = \{(x,y) \mid y - x = 0\}$

Let $U = \{(x,y) \mid y + x = 0\}$

Then W and U are both subspaces of $\mathbb{R}_2$ by Ex 1e.

The set $X = U \cup W$ is sketched $\longrightarrow$



It is clear that

$a = (1,1) \in W$ and $b = (1,-1) \in U$

but

$a + b = (2,0) \notin X$

Unions of subspaces need not be subspaces.