

Systems of Linear Equations: Section 8, Chapter 2

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Linear Systems

We will now focus on the problem of solving systems of linear equations with **real coefficients**. It is this problem that we encounter most often in practice.

A **system of m linear equations in n unknowns** will be denoted by

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

where for all $1 \leq i \leq m$ and $1 \leq j \leq n$, $a_{ij} \in \mathbb{R}$ and $b_i \in \mathbb{R}$. The numbers x_j for $1 \leq j \leq n$ are called the **unknowns**.

Solutions of Systems

We will say that

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$$

is the i -th equation in this system.

In most case, we consider the numbers a_{ij} and b_i to be fixed and we seek *solutions* of this system. We will say that the n -tuple $\mathbf{y} = (y_1, y_2, \cdots, y_n)$ is a **solution of this system** if \mathbf{y} satisfies:

$$a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n = b_1$$

$$a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n = b_2$$

$$\vdots$$

$$a_{m1}y_1 + a_{m2}y_2 + \cdots + a_{mn}y_n = b_m$$

i.e., \mathbf{y} is a solution if it satisfies the i -th equation for all $1 \leq i \leq m$.

The Coefficient Matrix

Corresponding to the system:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

\vdots

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

we can introduce the $m \times n$ **matrix** whose rows are the vectors

$$\mathbf{r}_1 = (a_{11}, a_{12}, \cdots, a_{1n}), \quad \mathbf{r}_2 = (a_{21}, a_{22}, \cdots, a_{2n}), \\ \cdots, \mathbf{r}_m = (a_{m1}, a_{m2}, \cdots, a_{mn}).$$

We often denote this by

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

and call \mathbf{A} the **coefficient matrix** corresponding to this system.

Matrices

Let \mathbf{A} be an $m \times n$ matrix:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

We have introduced matrices as an ordered collection of rows:

$$\{\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_m\} \subset \mathbb{R}_n$$

where for each $1 \leq i \leq m$

$$\mathbf{r}_i = (a_{i1}, a_{i2}, \cdots, a_{in})$$

We often call each \mathbf{r}_i a **row vector**.

Matrices

Let \mathbf{A} be an $m \times n$ matrix:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

We *could* have also introduced matrices as an ordered collection of *columns*, i.e. m -tuples:

$$\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\} \subset \mathbb{R}_m$$

where for each $1 \leq j \leq n$

$$\mathbf{c}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$$

Each m -tuple \mathbf{c}_j will be called a **column vector**.

Rows and Columns

It is useful to think of systems (and or matrices) in terms of both *rows* and *columns*.

To this end, given a matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

we will define the **row subspace** of \mathbf{A} as the subspace

$$S(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_m) \subset \mathbb{R}_n$$

i.e. the subspace of \mathbb{R}_n generated by the *rows* of \mathbf{A} .

Similarly, we will define the **column subspace** of \mathbf{A} as the subspace

$$S(\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n) \subset \mathbb{R}_m$$

i.e. the subspace of \mathbb{R}_m generated by the *columns* of \mathbf{A} .

Systems and Columns

Let us now fix numbers a_{ij} and b_i and consider the system

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

Denote by \mathbf{b} the column vector whose entries are the b_i . In this case, the system above can be written as:

$$x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n = \mathbf{b}$$

and therefore a **solution** corresponds to a specific linear combination of the columns which ultimately produces \mathbf{b} .

Homogeneous Systems

Let us now *just* fix a collection of numbers a_{ij} . We think of trying to solve this system with different collections of numbers b_i :

The system

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0$$

is said to be a **homogeneous** system of equations.

Written differently, this system is

$$x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n = \mathbf{0}$$

where we have taken $\mathbf{b} = \mathbf{0}$.

It is clear that $\mathbf{x} = \mathbf{0}$ is always a solution of a homogeneous system. It is called the **trivial solution**. Any solution \mathbf{x} of this system with $\mathbf{x} \neq \mathbf{0}$ is said to be a **non-trivial** solution.

Non-Homogeneous Systems

Again think that we have fixed a collection of numbers a_{ij} . Take a collection of numbers b_i with not all $b_i = 0$:

The system

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

is said to be a **non-homogeneous** system of equations.

Written differently, this system is

$$x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n = \mathbf{b}$$

where we have taken \mathbf{b} to be the column vector whose entries are the b_i .

It is clear that $\mathbf{x} = \mathbf{0}$ is never a solution of a non-homogeneous system.

The following is an important result.

Theorem

The non-homogeneous system

$$x_1 \mathbf{c}_1 + x_2 \mathbf{c}_2 + \cdots + x_n \mathbf{c}_n = \mathbf{b}$$

has a solution if and only if either of the following hold:

- i) \mathbf{b} belongs to the column space $S(\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n)$.*
- ii) $\dim(S(\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n)) = \dim(S(\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n, \mathbf{b}))$.*

Some Definitions

Here are two common definitions.

Definition

Let \mathbf{A} be an $m \times n$ matrix with columns $\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\}$. The **rank** of the matrix \mathbf{A} is said to be the dimension of the column space $S(\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n)$.

Definition

For any non-homogeneous system

$$x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \dots + x_n\mathbf{c}_n = \mathbf{b},$$

the matrix with columns $\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n, \mathbf{b}\}$ is called the **augmented matrix** corresponding to this system.

A Re-Statement

The previous result can be restated as follows:

Theorem

The non-homogeneous system

$$x_1 \mathbf{c}_1 + x_2 \mathbf{c}_2 + \cdots + x_n \mathbf{c}_n = \mathbf{b}$$

has a solution if and only if the rank of the coefficient matrix is equal to the rank of the corresponding augmented matrix.

This provides us a way of determining whether or not there is a solution of a given non-homogenous system.

We will now discuss the form of *all* possible solutions.

Theorem

Suppose that the non-homogeneous system

$$x_1 \mathbf{c}_1 + x_2 \mathbf{c}_2 + \cdots + x_n \mathbf{c}_n = \mathbf{b}$$

has a solution \mathbf{x} . Consider the associated homogeneous system

$$x_1 \mathbf{c}_1 + x_2 \mathbf{c}_2 + \cdots + x_n \mathbf{c}_n = \mathbf{0}$$

- i) Given any solution \mathbf{x}_0 of the homogeneous equation, $\mathbf{y} = \mathbf{x} + \mathbf{x}_0$ is a solution of the non-homogeneous equation.*
- ii) Any solution \mathbf{y} of the non-homogeneous equation can be written as $\mathbf{y} = \mathbf{x} + \mathbf{x}_0$ for some \mathbf{x}_0 solving the homogeneous equation.*

Example 1

Ex. 1 Test the following system of equations for solvability and find a solution if there is one.

$$\begin{aligned}x_1 + 2x_2 - 3x_3 + x_4 &= 1 \\x_1 + x_2 + x_3 + x_4 &= 0\end{aligned}$$

Sol. First, let's label the pieces. Here

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -3 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

is the coefficient matrix of the system with columns

$$\mathbf{c}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{c}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \mathbf{c}_3 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}, \quad \mathbf{c}_4 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The system can be re-written as

$$x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + x_3\mathbf{c}_3 + x_4\mathbf{c}_4 = \mathbf{b} \quad \text{with} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Example 1 (cont.)

To determine whether or not a solution exists, we can compare the rank of the coefficient matrix with the rank of the augmented matrix:

$$\mathbf{B} = \begin{pmatrix} 1 & 2 & -3 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Clearly,

$$\text{rank}(\mathbf{A}) = \dim(S(\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4))$$

We can calculate this dimension by performing elementary row operations to

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \\ -3 & 1 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & -1 \\ 0 & 4 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

The dimension is 2, and so $\text{rank}(\mathbf{A}) = 2$.

Example 1 (cont.)

It is clear that $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4\} \subset \mathbb{R}_2$. Since we just showed that

$$\dim(S(\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4)) = 2 \quad \Rightarrow \quad S(\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4) = \mathbb{R}_2.$$

For this reason, it is clear that

$$\text{rank}(\mathbf{B}) = \dim(S(\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4, \mathbf{b})) = 2$$

as well. This shows that

$$\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{B})$$

and hence, by the previous result, there is a solution.

Example 1 (cont.)

We will now find a solution to this system. To do so, we perform elementary row operations to the augmented matrix

$$\mathbf{B} = \begin{pmatrix} 1 & 2 & -3 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

and put it in echelon form. As we discussed in class, this preserves the system of equations. In other words, any solution corresponding to \mathbf{B} is also a solution corresponding to \mathbf{B}' where \mathbf{B}' is row equivalent to \mathbf{B} (and vice versa!).

Clearly

$$\mathbf{B} \sim \begin{pmatrix} 1 & 2 & -3 & 1 & 1 \\ 0 & -1 & 4 & 0 & -1 \end{pmatrix}$$

Example 1 (cont.)

The row equivalent matrix

$$\mathbf{B}' = \begin{pmatrix} 1 & 2 & -3 & 1 & 1 \\ 0 & -1 & 4 & 0 & -1 \end{pmatrix}$$

corresponds to the system

$$\begin{aligned}x_1 + 2x_2 - 3x_3 + x_4 &= 1 \\ -x_2 + 4x_3 &= -1\end{aligned}$$

We need only find one solution. Let's set $x_3 = 0$. In this case, the second equation implies that $x_2 = 1$. Plugging this into the first equation, we find that

$$x_1 + x_4 = -1$$

Take $x_4 = 0$ and $x_1 = -1$. We have found a solution

$$\mathbf{x} = (-1, 1, 0, 0).$$

One easily checks that this also solves the original system.