MATH 523
FINAL

FALL 2009

Due Tuesday December 15th, 2009 before noon.

(1) Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Let $f : X \to [0, \infty)$ be a measurable function. Denote by

$$F(t) = \mu\left(\{x \in X : f(x) > t\}\right) .$$

Prove that

$$\int_X f(x) d\mu(x) = \int_0^\infty F(t) dt \,,$$

where the integral on the right-hand side above is with respect to Lebesgue measure on $[0, \infty)$. Note: This problem shows that integration with respect to an "abstract measure" can be done using the Lebesgue integral.

(2) Verify the following identity. For any $s > 0$,

$$\int_0^\infty e^{-sx} \frac{\sin^2(x)}{x} dx = \frac{1}{4} \ln\left(1 + \frac{4}{s^2}\right) .$$

It may be useful to integrate the function $f_s(x, y) = e^{-sx} \sin(2xy)$ with respect to x and y. Here the integral above is with respect to Lebesgue measure on $[0, \infty)$.

(3) Evaluate the following limit:

$$\lim_{n\to\infty} \int_0^1 \frac{nx \ln(x)}{1 + n^2 x^2} dx \,.$$

Here the integral above is with respect to Lebesgue measure on $[0, 1]$.

(4) Let $(X_i, \mathcal{M}_i, \mu_i)$ be σ-finite measure spaces for $i = 1, 2, 3$. Using the methods discussed in class, construct

$$\nu_1 = (\mu_1 \times \mu_2 \times \mu_3)$$

and

$$\nu_2 = (\mu_1 \times \mu_2) \times \mu_3 \,.$$

Prove that $\nu_1 = \nu_2$.

(5) Verify the following equality: For all $a > 0$,

$$\int_{-\infty}^{\infty} e^{-x^2} \cos(ax)\, dx = \sqrt{\pi} e^{-\frac{a^2}{4}} \,.$$

The one-dimensional version of Proposition 2.53 may be of use in your argument. Here the integral above is with respect to Lebesgue measure on $\mathbb{R}$