MATH 523
MIDTERM

FALL 2009

Due Monday October 26th, 2009.

(1) Let X be a non-empty set equipped with the σ-algebra $\mathcal{M} = \mathcal{P}(X)$. For any function $f : X \to [0, \infty]$, show that $\mu_f : \mathcal{M} \to [0, \infty]$ given by

$$\mu_f(E) = \sum_{x \in E} f(x) \quad \text{for all } E \in \mathcal{M}\,,$$

is a measure.

(2) Do problem 21 in Chapter 1. You may use the results of any previous homework problem without proof, however, you should reference them clearly in your argument.

(3) Complete the last step in the proof of Proposition 1.15. Specifically, write a complete argument demonstrating that: if $\{I_j\}_{j=1}^{\infty}$ is a disjoint sequence of h-intervals with

$$\bigcup_{j=1}^{\infty} I_j = (a, \infty)\,,$$

then

$$\mu_0\left(\bigcup_{j=1}^{\infty} I_j\right) = \sum_{j=1}^{\infty} \mu_0(I_j).$$

Explain clearly all inequalities used, in particular, those used in the text.

(4) Show that any increasing function $f : \mathbb{R} \to \mathbb{R}$ has at most countably many jump-discontinuities. (An increasing function $f : \mathbb{R} \to \mathbb{R}$ is said to have a jump discontinuity at a if $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$.) Use this result to prove that the Cantor-Lebesgue function is continuous.

(5) Let X be a non-empty set equipped with a σ-algebra $\mathcal{M}$. Suppose $A, B \in \mathcal{M}$ and $X = A \cup B$. Prove that any function $f : X \to \mathbb{C}$ is measurable if and only if f is measurable on A and on B.