MATH 523 B
FINAL

SPRING 2010

The following assignment is to be turned in on
Friday, May 7th, 2010.

(1) Let $(\mathbb{R}, \mathcal{M}, \mu)$ be a measure space. Consider the function $\phi : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ given by

$$\phi[t] = \log\left(\int e^{tx} d\mu(x)\right) .$$

Prove that ϕ is convex in the sense that for each $0 \leq \lambda \leq 1$ and any $t_1, t_2 \in \mathbb{R}$

$$\phi[\lambda t_1 + (1-\lambda)t_2] \leq \lambda\phi[t_1] + (1-\lambda)\phi[t_2] .$$

(2) This question concerns limits of norms in L^p spaces. You may wish to consult problems 7 and 8 in Section 6 of the book.

a) Let $0 < p < \infty$ and take $f \in L^p \cap L^\infty$. Prove that

$$\lim_{q\to\infty} \|f\|_q = \|f\|_\infty$$

where q is any number $q > p$.

b) Let $0 < p < \infty$ and suppose that $f \in L^p(X, d\mu)$ where $\mu(X) = 1$. Prove that

$$\lim_{q\to 0} \|f\|_q = \exp\left(\int \log(|f|) d\mu\right) ,$$

where here q is any number $0 < q < p$.

(3) Do problem 29 on page 196.

(4) Do problem 18 on page 255.

(5) Consider the function $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = \begin{cases} 1 - |x| & \text{if } |x| \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Use the Poisson summation formula to prove that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+\alpha)^2} = \frac{\pi^2}{\sin^2(\pi\alpha)}$$

is valid for all $\alpha \in \mathbb{R} \setminus \mathbb{Z}$.

Hint: You may first want to verify that

$$\hat{g}(\xi) = \begin{cases} 1 & \text{if } \xi = 0\,, \\ \frac{\sin^2(\pi\xi)}{(\pi\xi)^2} & \text{otherwise.} \end{cases}$$

(6) (Bonus) The following problem is for extra credit; you will not be penalized if you do not present the solution. Do problem #30 on page 196.