# MATH 528 A
# MIDTERM

FALL 2010

Due , 2010.

(1) Let $X$ be a vector space over $\mathbb{C}$. A mapping $s : X \times X \to \mathbb{K}$ is called a *sesquilinear form* on $X$ if:
for all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{C}$,

$$s(x, \alpha y + \beta z) = \alpha s(x, y) + \beta s(x, z)$$

and

$$s(\alpha x + \beta y, z) = \overline{\alpha} s(x, z) + \overline{\beta} s(y, z) \, .$$

Given such a sesquilinear form on $X$, the mapping $q : X \to \mathbb{C}$ defined by setting

$$q(x) = s(x, x) \quad \text{for each } x \in X \, ,$$

is called the *quadratic form* induced by $s$. Prove that each sesquilinear form is uniquely determined by its quadratic form.

(2) Let $\mathcal{H}$ be a Hilbert space. For any integer $n \geq 2$ and any collection $\{f_j\}_{j=1}^n \subset \mathcal{H}$, the *Gram determinant* of $\{f_j\}_{j=1}^n$ is defined by setting

$$D(f_1, f_2, \cdots, f_n) = \det\left(\langle f_j, f_k \rangle\right) \, ,$$

i.e., the determinant of the matrix $A$ whose entries $a_{jk} = \langle f_j, f_k \rangle$ are the corresponding inner-products. Prove that $D(f_1, f_2, \cdots, f_n) \geq 0$ with equality if and only if the collection $\{f_j\}_{j=1}^n$ are linearly dependent. Note: The case of $n = 1$ is silly; I didn't even include it. The case $n = 2$ should be familiar. The rest follows.

(3) Let $X$ be a normed space over $\mathbb{R}$. For any bounded subset $M \subset X$, define the *support function* $S_M : X^* \to \mathbb{R}$ by setting

$$S_M(f) = \sup_{x \in M} f(x) \, .$$

Prove that $S_M$ is:

a) *Sub-additive*: i.e., for all $f, g \in X^*$, $S_M(f + g) \leq S_M(f) + S_M(g)$.

b) *Monotonic*: i.e., for $M \subset N$, $S_M(f) \leq S_N(f)$ and *Additive*: i.e., $S_{N+M} = S_N + S_M$.

c) Show that $S_{\overline{M}} = S_M$ where $\overline{M}$ is the closure of $M$.

d) Show that $S_{L(M)} = S_M$ where $L(M)$ is the set of all finite, convex combinations of elements of $M$.

(4) Let $X$ be a vector space over $\mathbb{C}$ with a metric. Let $F \subset X$ be relatively compact. Prove Arzelá, i.e. the analogue of Theorem 4.3.1, for any set $M \subset C(F)$.

(5) Consider the Hilbert space $\mathcal{H} = L^2([0,1])$. Let $A : \mathcal{H} \to \mathcal{H}$ be given by

$$[Af](t) = t \cdot f(t) \quad \text{for all } f \in \mathcal{H}\,.$$

Prove that $\sigma(A) = [0.1]$. In fact, show that $\sigma(A) = [0,1] = \sigma_c(A)$.