Due December 11, 2012.

(1) Do problem 2.20 in the book.

(2) Let $A$ be a self-adjoint operator on a separable, complex Hilbert space $\mathcal{H}$. Let $P_A$ be the corresponding projection-valued measure (whose existence is guaranteed by the spectral theorem). Let $\psi \in \mathcal{H}$ and $N \subset \mathbb{R}$ be a Borel set. Prove that

$$P_A(N)\psi = \psi \quad \text{if and only if} \quad \mu_\psi(\mathbb{R} \setminus N) = 0.$$ 

Here $\mu_\psi$ is the Borel measure given by $\mu_\psi(\Omega) = \langle \psi, P_A(\Omega)\psi \rangle$.

(3) Prove Lemma 3.12 in full detail - I discussed much of this in class, but write up your own version. You may assume any lemma, theorem, or homework problem prior to Lemma 3.12 in the book. Prove or disprove (via a counter-example) the following statement: Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. $f(A)$ is closed for any $A \subseteq \mathbb{R}$ closed.
(4) Let $A$ be a self-adjoint operator on a separable, complex Hilbert space $\mathcal{H}$. Let $M$ be a reducing subspace for $A$. Define an operator $A_M$ by setting $D(A_M) = M \cap D(A)$ and $A_M \psi = A\psi$ for all $\psi \in D(A_M)$. Prove that $M^\perp$ is also a reducing subspace for $A$. Show that $A_M$ and $A_M^\perp$ are both self-adjoint on $M$ and $M^\perp$ respectively. Show that $\sigma(A) = \sigma(A_M) \cup \sigma(A_M^\perp)$. 