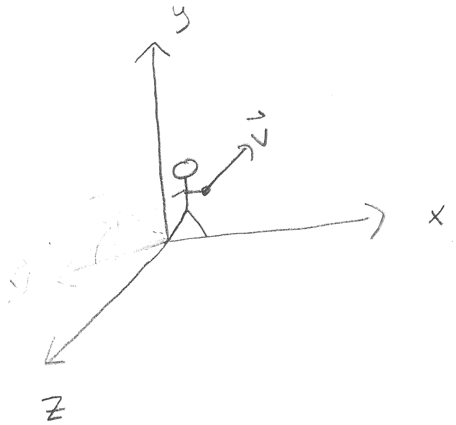
# Motivation




If I have a vector $\vec{v} \in \mathbb{R}^3$ and I rotate an angle $\theta$ about the y-axis, I can describe the way $\vec{v}$ changes via a matrix

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \cos\theta + v_3 \sin\theta \\ v_2 \\ -v_1 \sin\theta + v_3 \cos\theta \end{bmatrix}$$

But what if $\vec{v}$ doesn't naturally live in $\mathbb{R}^3$? What if I instead have some $\vec{w} \in \mathbb{C}^n$? $\underbrace{\text{How does } \vec{w} \text{ transform under rotations?}}$

These notes are my attempt at a motivated answer to this question.

# Infinitesimal Rotations

**Note** $R_x(\frac{\pi}{2}) \circ R_y(\frac{\pi}{2}) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$$\neq$$

$R_y(\frac{\pi}{2}) \circ R_x(\frac{\pi}{2}) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$

I.e. the sequence in which you perform rotations is important.

We can rewrite $R_x(\theta) = \overbrace{R_x(\frac{\theta}{n}) \circ R_x(\frac{\theta}{n}) \ldots \circ R_x(\frac{\theta}{n})}^{n} = \left[R_x(\frac{\theta}{n})\right]^n$

i.e. a rotation about x-axis is a sequence of small rotations.

We approximate $R_x(\frac{\theta}{n}) \simeq I + \frac{dR_x}{d\theta} \cdot \frac{\theta}{n}$

$$\Rightarrow R_x(\theta) = \lim_{n \to \infty} \left[ I + \frac{dR_x}{d\theta} \cdot \frac{\theta}{n} \right]^n = e^{\frac{dR_x}{d\theta} \cdot \theta}$$

So we will look at how $\vec{w} \in \mathbb{C}^n$ transforms under infinitesimal rotations.

## Properties of $\rho$ $\quad so(3) := \{L_x, L_y, L_z\}$

Assuming linearity, we are thus interested in maps $\rho: so(3) \to Mat(\mathbb{C}^n)$

where the induced rotation is given by $e^{\theta \rho(L_x)}$

What properties should $\rho$ satisfy? $\rho(R_1 \circ R_2) = \rho(R_1) \circ \rho(R_2) \Rightarrow$ homomorphism

### ① Preservation of Lie Bracket $\quad [A, B] := AB - BA$

Let's look at the difference between

$R_x(\Delta x) \circ R_y(\Delta y)$ vs $R_y(\Delta y) \circ R_x(\Delta x)$, i.e.

how different is $R_x(\Delta x) \circ R_y(\Delta y) \circ R_x^{-1}(\Delta x) \circ R_y^{-1}(\Delta y)$

Approximating we get

$[I + L_x \Delta x] \circ [I + L_y \Delta y] \circ [I - L_x \Delta x] \circ [I - L_y \Delta y]$

$\simeq I + [L_x, L_y] \Delta x \Delta y$ + higher order terms..

$\Rightarrow [\,,\,]$ encodes information about how rotations act,

I thus insist that $\rho$ preserves $[\,,\,]$.

# Properties of $\rho$

② Unitarity

If I have a closed system sitting on a table & I rotate the entire table, I would like if the "observables don't change."

Here the observables arise out of inner products.
Hence I insist that, if $\vec{v}, \vec{w} \in \mathbb{C}^n$ and $\mathcal{U}$ is my rotation matrix,
$\langle \vec{v}, \vec{w} \rangle = \langle \mathcal{U}\vec{v}, \mathcal{U}\vec{w} \rangle$, i.e. $\mathcal{U}$ is unitary.

## Properties of $\rho$

### 3 Irreducibility

Let $G$ denote our induced space of unitary matrices acting on $\mathbb{C}^n$

$V \leq \mathbb{C}^n$ is called <u>invariant</u> if $g\vec{v} \in V \;\; \forall g \in G, \vec{v} \in V$

$V$ is called <u>irreducible</u> if $\nexists$ nontrivial subspace of $V$ that is also <u>invariant</u>.

**FACT** $\mathbb{C}^n = V_1 \oplus \ldots \oplus V_k$, where each $V_j$ is irreducible AND pairwise orthogonal

proof in A1

**FACT** $\varphi(SO(3))$ acts irreducibly on each $V_j$

proof in A2

Hence $\varphi \cong \rho_1 \oplus \ldots \oplus \rho_k$

Thus it suffices to assume $\rho$ is irreducible

Consequential Properties of $\rho$

unitarity $\Rightarrow$ $J_x := i\rho(L_x)$ hermitian

$[J_x, J_y] = iJ_z \Rightarrow tr([J_x, J_y]) = 0 = i\,tr(J_z)$

$\Rightarrow$ $J_x$ is traceless

## Uniqueness

follows from unitarity assumption

assume $\rho$ satisfies our conditions

$J_x := i\rho(L_x) \Rightarrow J_x$ is hermitian, $[J_x, J_y] = iJ_z$

Note if $[N, X] = cX \Rightarrow NXv_j = (\lambda_j + c)Xv_j$

where $\lambda_j, v_j$ are eigenvalue/vectors of $N$

We look for

$$[J_z, \alpha J_x + \beta J_y + \gamma J_z] = c(\alpha J_x + \beta J_y + \gamma J_z)$$

$$\parallel$$

$$i\alpha J_y - i\beta J_x = c(\ldots$$

$$i\alpha = c\beta$$

$$-i\beta = c\alpha = -\frac{i^2\alpha}{c} \Rightarrow \boxed{c = \pm 1}$$

let $J_\pm := J_x \pm iJ_y \Rightarrow [J_z, J_\pm] = \pm J_\pm$, $[J_+, J_-] = 2J_z$

let $\lambda_n$ be (a) largest eigenvalue of $J_z \Rightarrow \boxed{J_+ \vec{v}_{\lambda_n} = 0}$

let $v_{\lambda_k} := (J_-)^k v_{\lambda_n}$,

FACT By irreducibility $\{v_{\lambda_k}\}$ forms a basis for $\mathbb{C}^n$

Uniqueness

FACT

Computation in (A4)

$$\boxed{\lambda_n = \frac{n-1}{2}} \leftarrow \text{called "spin"}$$

i.e., $\lambda_n$ is determined by dimension of $\mathbb{C}^n$ entirely!!!

Hence given two mcps $\rho, \tilde{\rho}$, the eigenvectors of $J_z$

induce a natural isomorphism!

$SU(2)$ $V_{\frac{1}{2}} = \mathbb{C}^2$, spin $\frac{1}{2}$

$J_x = i\rho(L_x)$, note $\vec{\sigma}$ basis for hermitian traceless matrices

$J_x = \vec{r}_x \cdot \vec{\sigma}$, note $(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) = (\vec{a}\cdot\vec{b}) + i(\vec{a}\times\vec{b})\cdot\vec{\sigma}$

$\Rightarrow [J_x, J_y] = i\,2(\vec{r}_x \times \vec{r}_y)\cdot\vec{\sigma} = iJ_z \Rightarrow 2\|r_x\|\|r_y\|\sin\theta = \|r_z\|$

$[J_y, J_z] = i4(\vec{r}_y \times (\vec{r}_x \times \vec{r}_y))\cdot\vec{\sigma} = iJ_x \Rightarrow 4\|r_y\|^2\|r_x\|\sin\theta = \|r_x\|$

$\Rightarrow 4\|r_y\|^2\sin\theta = 1$ & $\|r_x\| = \|r_y\| = \|r_z\|$

$\Rightarrow 2\|r_y\|\sin\theta = 1 \Rightarrow \boxed{\|r_y\| = \frac{1}{2}} \Rightarrow \sin\theta = 1$

Result $J_x = \frac{1}{2}(\vec{r}_x \cdot \vec{\sigma})$, $\|r_x\| = 1$, $\vec{r}_x \cdot \vec{r}_y = 0$

renaming $\vec{r}_x$

Rotations about x-axis given by:

$$e^{\theta i(\frac{1}{2}\vec{r}_x\cdot\vec{\sigma})} = \cos\left(\frac{\theta}{2}\right) + i(\vec{r}_x \cdot \vec{\sigma})\sin\left(\frac{\theta}{2}\right)$$

note $\boxed{e^{i2\pi J_x} = -1}$

Note

$J$ traceless hermitian, $\Rightarrow e^{iJ} = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$, $|\alpha|^2 + |\beta|^2 = 1$

This space denoted by $SU(2)$

"Rotations" in $\mathbb{C}^2$ given by elements of $SU(2)$

Computing Spin 1 Matrices $V_{\underline{1}} = \mathbb{C}^3 \ldots$

Much harder to do by hand, instead take an indirect route

We form tensor product $V_{\frac{1}{2}} \otimes W_{\frac{1}{2}}$, claim $\cong V_1 \oplus V_0$

Infinitesimal "Rotations" $= \frac{d}{d\theta}\left(Q^v(\theta) \otimes Q^w(\theta)\right)\Big|_{\theta=0} = J^v \otimes I + I \otimes J^w$

$\underline{J := J^v \otimes I + I \otimes J^w}$, same commutation relations only

Using basis from uniqueness, form basis $V_\alpha \otimes w_\beta$

$\lambda := \frac{1}{2}$

where $J_3 (v_\alpha \otimes w_\beta) = \left[(\lambda - \alpha) + (\lambda - \beta)\right] v_\alpha \otimes w_\beta$

$J_-(v_\alpha \otimes w_\beta) = v_{\alpha+1} \otimes w_\beta + v_\alpha \otimes w_{\beta+1}$

$J_-^N (v_\alpha \otimes w_\beta) = \sum_{\mu=0}^{N} \binom{N}{\mu} v_{\alpha+\mu} \otimes w_{\beta+N-\mu}$

In our case, we only have $v_0, v_1, v_2 = 0 \ldots$

$J_-^N(v_0 \otimes w_0) = 0$ when $N > 2$, $J_-$ induces a 3 dimensional space of eigenvectors of $J_3$, denoted $V_1$

$\vec{J}$ acts irreducibly on $V_1$

$V_{\frac{1}{2}} \otimes W_{\frac{1}{2}} = V_1 \oplus (V_1)^\perp$

## Computing Spin 1 Matrices

Note either $V_0 \otimes w_1$ or $V_1 \otimes w_0$ & $V_1$

Why? They have same eigenvalue w.r.t $J_3$, and are linearly independent

Sketch of proof

$\alpha^j v_j = w$, $v_j$ distinct eigenvalues w.r.t. to $T$,

$T(\alpha^j v_j) = \alpha^j \lambda_j v_j = \lambda_1 w \Rightarrow \lambda_1 \alpha^j v_j = \alpha^j \lambda_j v_j$

$\Rightarrow \quad \lambda_1 \alpha^j = \lambda_j \alpha^j \quad \forall j$

Case 1) $\lambda_1 = 0 \Rightarrow \lambda_2, \ldots \lambda_n \neq 0 \Rightarrow \alpha^1 v_1 = w$ $\divideontimes$ not true for us

Case 2) $\lambda_1 \neq 0$, at most 1 $\lambda_j = 0$, call it $\lambda_2$

$\alpha^1 v_1 + \alpha^2 v_2 = w$

$\lambda_1 \alpha^1 v_1 + 0 = \lambda_1 w$ $\divideontimes$

Assume $V_0 \otimes w_1$ & $V_1$, $J_3$ hermitian $\Rightarrow$ eigenvectors w/ distinct eigenvalues orthogonal to $V_0 \otimes w_1$

Using gram-schmidt, the basis for $(V_1)^{\perp}$ is still an eigenvector of $J_3$.

$$\Rightarrow \boxed{V_{\frac{1}{2}} \otimes V_{\frac{1}{2}} \simeq V_1 \oplus V_0}$$

General proof follows this format, $V_\ell \otimes V_m \simeq V_{\ell+m} \oplus V_{\ell+m-1} \oplus \ldots \oplus V_{\ell-m}$

A1

let $W \leq \mathbb{C}^n$ irreducible, $W^\perp$ orthogonal complement, $g \in G$, $x \in W^\perp$, $y \in W$

$\langle gx, y\rangle = \langle x, g^\dagger y\rangle = 0 \Rightarrow$ $W^\perp$ invariant, proof follows by induction

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$X \in \rho(\mathfrak{su}(3))$, $\mathbb{C}^n = W \oplus W^\perp$, $Xy = a + b$, $y \in W$, $a \in W$, $b \in W^\perp$.

$$0 = \langle e^{tX} y, b \rangle = \langle \sum \frac{t^n X^n}{n!} y, b \rangle = \langle y, b \rangle + t \langle Xy, b \rangle + \text{higher order terms}$$

taking derivative at $t = 0 \Rightarrow$

$$0 = \langle Xy, b \rangle = \cancel{\langle a, b \rangle}^{\to 0} + \langle b, b \rangle \Rightarrow b = 0$$

$\Rightarrow$ $\rho$ acts invariantly on $W$

assume $W = V_1 \oplus V_2$, $\rho(\mathfrak{so}(3)) V_j = V_j$

by linearity, $G V_j = V_j$ $\divideontimes$,

Conclude $\rho$ acts irreducibly.

A4

$$0 = J_+ J_- v_{\lambda_n - (n-1)} = J_+ J_- ( J_-^{(n-1)} v_{\lambda n})$$

$$= ( [J_+, J_-] + J_- J_+ )( J_-^{(n-1)} v_{\lambda n})$$

$$= ( 2 J_z J_-^{(n-1)} + J_- ( J_+ J_- ) J_-^{(n-2)} ) v_{\lambda n}$$

$$= \sum_{k=0}^{n-1} J_-^k (2 J_z) J_-^{(n-1)-k} v_{\lambda n}$$

$$= 2 \sum_{k=0}^{n-1} J_-^k ( \lambda_n - (n-1-k)) J^{(n-1)-k} v_{\lambda n}$$

$$= 2 \sum_{k=0}^{n-1} ( \lambda_n - (n-1-k)) \cdot J_-^{(n-1)} v_{\lambda n}$$

$$\Rightarrow 2\lambda_n - n + 1 = 0 \Rightarrow \boxed{\lambda_n = \frac{n-1}{2}}$$

## Tensor Products A5

let $V_\ell, V_m$ $2\ell+1$, $2m+1$ dimension w/ irreducible rep, $\ell \geq m$

**Claim** $V_\ell \otimes V_m = V_{\ell+m} \oplus V_{\ell+m-1} \oplus \cdots \oplus V_{\ell-m}$

pf)

$$J^\pm := J_\ell^\pm \otimes I + I \otimes J_m^\pm$$

$$J^3 := J_\ell^3 \otimes I + I \otimes J_m^3$$

$$J^3 (v_\alpha \otimes w_\beta) = [(\ell-\alpha) + (m-\beta)] v_\alpha \otimes w_\beta$$

$$(J_-)^N v_\alpha \otimes w_\beta = \sum_{\mu=0}^{N} \binom{N}{\mu} v_{\alpha+\mu} \otimes w_{\beta+N-\mu} = 0 \quad \text{for } \alpha+\mu \geq 2\ell$$

$$\beta+N-\mu \geq 2m$$

$$\Rightarrow N \geq 2(\ell+m) - \alpha - \beta$$

$(J_-)$ acts iteratively to yield $2(\ell+m) - \alpha - \beta + 1$ space of eigenvectors of $J^3$

$\Rightarrow$ $v_0 \otimes w_0$ induces $2(\ell+m)+1$ space denote $V_{\ell+m}$

$\Rightarrow$ $V_\ell \otimes V_m = V_{\ell+m} \oplus (V_{\ell+m})^\perp$

## Tensor Products P5



Either $V_1 \otimes w_0$ or $V_0 \otimes w_1$ & $V_{l+m}$ ← Same eigenvalue b/c

The one that isn't induces $2(l+m-1)+1$ dimensional space, $V_{l+m-1}$,

using gram-schmidt + $J^3$ being hermitian, we construct space of

eigenvectors orthogonal to $V_{l+m}$, of dimension $2(l+m-1)+1$

$$\Rightarrow V_l \otimes V_m = V_{l+m} \oplus V_{l+m-1} \oplus (V_{l+m-1})^{\perp}$$

Assume

$$V_l \otimes V_m = V_{l+m} \oplus \dots \oplus (V_{l+m-j})^{+}$$

need $V_\alpha \otimes w_\beta$ & $(V_{l+m} \oplus \dots \oplus V_{l+m-j})$ s.t.

$$J^3(V_\alpha \otimes w_\beta) = \left((l-\alpha)+(m-\beta)\right) V_\alpha \otimes w_\beta = \left(l+m-(j+1)\right) V_\alpha \otimes w_\beta$$

$\Rightarrow j+1 = \alpha+\beta$, note $j+1 \leq 2m \Rightarrow j+2$ choices for $\alpha+\beta$

$$\Rightarrow V_l \otimes V_m = V_{l+m} \oplus V_{l+m-1} \oplus \dots \oplus V_{l-m} \oplus \underline{\underline{W}}$$

$$l \cdot m = \underbrace{\sum_{j=0}^{2m} (l+m-j)}_{= \, l \cdot m} + \dim W \Rightarrow \dim W = 0$$

$$\Rightarrow \boxed{V_l \otimes V_m = \bigoplus_{j=0}^{2m} V_{l+m-j}}$$