Lecture 11

On Entanglement for Bipartite Systems

Consider two quantum systems represented by finite dimensional (non-zero) complex Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. The composite system $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ is often referred to as a bipartite system.

A state ω on $\mathcal{A} = B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is said to be a <u>product state</u> if there are sates ω_1 on $\mathcal{A}_1 = B(\mathcal{H}_1)$ and ω_2 on $\mathcal{A}_2 = B(\mathcal{H}_2)$ for which

$$\omega(A \otimes B) = \omega_1(A) \cdot \omega_1(B) \quad \text{for all } A \in B(\mathcal{H}_1) \text{ and } B \in B(\mathcal{H}_2).$$

<u>Ex</u> In the finite dimensional setting considered above, let $\rho_i \in B(\mathcal{H}_i)$ for $i=1,2$ be density matrices.
One easily checks that $\rho = \rho_1 \otimes \rho_2 \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is a density matrix and the corresponding state:

$$\omega_\rho(A) = \mathrm{Tr}[\rho A] \quad \text{for all } A \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$$

is a product state.

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Definition A state ω on $\mathcal{A} = \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is said to be separable if and only if it is a convex combination of product states. Any state ω on $\mathcal{A} = \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ that is not separable is said to be entangled.

It is easy to check that the set of separable states on $\mathcal{A} = \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is convex.

Lemma Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be finite dimensional complex Hilbert spaces. If ω is a separable state on $\mathcal{A} = \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, then the corresponding density matrix can be written as a convex combination of pure product states.

proof:

Let ω be a separable state on $\mathcal{A} = \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Then

$$\omega(A \otimes B) = \sum_{i=1}^{n} t_i\, \omega_i^{(1)}(A) \cdot \omega_i^{(2)}(B) \quad \text{for all } A \in \mathcal{B}(\mathcal{H}_1) \text{ and } B \in \mathcal{B}(\mathcal{H}_2)$$

where $t_i \geq 0$ and $\sum_{i=1}^{n} t_i = 1$. (Here $\omega_i^{(1)}$ and $\omega_i^{(2)}$ are states on $\mathcal{A}_1 = \mathcal{B}(\mathcal{H}_1)$ and $\mathcal{A}_2 = \mathcal{B}(\mathcal{H}_2)$ respectively.)

One easily checks that the corresponding density matrices satisfy

$$\rho = \sum_{i=1}^{n} t_i\, \rho_i^{(1)} \otimes \rho_i^{(2)} .$$

An application of the spectral theorem shows that: (3)

For each $1 \le i \le n$,

$$\rho_i^{(1)} = \sum_{j=1}^{\dim(H_1)} \lambda_j^i \, |\phi_j^i\rangle\langle\phi_j^i| \quad \text{where } \lambda_j^i \ge 0 \text{ and } \sum_{j=1}^{\dim(H_1)} \lambda_j^i = 1$$

and

$$\rho_i^{(2)} = \sum_{k=1}^{\dim(H_2)} \mu_k^i \, |\psi_k^i\rangle\langle\psi_k^i| \quad \text{where } \mu_k^i \ge 0 \text{ and } \sum_{k=1}^{\dim(H_2)} \mu_k^i = 1$$

since these are all density matrices.

In this case,

$$\rho = \sum_{i=1}^{n} t_i \, \rho_i^{(1)} \otimes \rho_i^{(2)} = \sum_{i=1}^{n} t_i \sum_{j=1}^{\dim(H_1)} \lambda_j^i \sum_{k=1}^{\dim(H_2)} \mu_k^i \left(|\phi_j^i\rangle\langle\phi_j^i|\right) \otimes \left(|\psi_k^i\rangle\langle\psi_k^i|\right)$$

Noting that

- $\left(|\phi_j^i\rangle\langle\phi_j^i|\right) \otimes \left(|\psi_k^i\rangle\langle\psi_k^i|\right) = |\phi_j^i \otimes \psi_k^i\rangle\langle\phi_j^i \otimes \psi_k^i|$

and moreover

- $$\sum_{i=1}^{n} t_i \sum_{j=1}^{\dim(H_1)} \lambda_j^i \sum_{k=1}^{\dim(H_2)} \mu_k^i = \sum_{i=1}^{n} t_i \sum_{j=1}^{\dim(H_1)} \lambda_j^i \, (1)$$
$$= \sum_{i=1}^{n} t_i \cdot (1) = 1$$

we are done.

As a consequence of this lemma we conclude:

The extreme points of the set of separable states are pure product states.

In particular, a pure state is separable if and only if it is a product state.

<u>Measuring Entanglement in Bipartite Systems</u> (again in finite dimensions!)

The most common measure of entanglement in a bipartite system is the Entanglement entropy.

For a pure state, we define entanglement entropy as follows:

Let ω be a pure state on $\mathcal{A} = \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. In this case,

$$\omega(A) = \mathrm{Tr}[\rho A] \quad \text{for all } A \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$$

and $\quad \rho = |\psi\rangle\langle\psi| \quad$ for some unit vector $\psi \in \mathcal{H}_1 \otimes \mathcal{H}_2$.

In this case, the entanglement entropy of ρ, which we denote by $S_E(\rho)$, is defined by

$$S_E(\rho) = S(\rho_1) = -\mathrm{Tr}[\rho_1 \ln(\rho_1)]$$

where $\rho_1 = \mathrm{Tr}_{\mathcal{H}_2}[\rho]$ is the partial trace of ρ.

In words, the entanglement entropy of a pure state ρ is defined to be the von-Neumann entropy of the restriction of ρ to the 1^{st} subsystem.

It is natural to ask: Why restrict to the 1^{st} subsystem? Why not use the 2^{nd} subsystem?

A goal of this lecture is to answer this question. We will show that:

$$(*) \qquad S(\rho_1) = S(\rho_2) \qquad \text{where } \rho_2 = \mathrm{Tr}_{\mathcal{H}_1}(\rho)$$

and so it does not matter which restriction (i.e. partial trace) you use.

Note: Since the entanglement entropy of a pure state is defined in terms of the von-Neumann entropy it is clear that

$$0 \leq S_E(\rho) \leq \min\left(\ln(\dim(\mathcal{H}_1)), \ln(\dim(\mathcal{H}_2))\right)$$

from the previous lecture. Here, for the upper bound, we have used $(*)$.

For a mixed state, the entanglement entropy is defined differently.

Let $\rho \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ be the density matrix associated to a mixed state. The entanglement entropy of ρ is defined as:

$$S_E(\rho) = \inf \left\{ \sum_{i=1}^{n} t_i S_E(\rho_i) \;\middle|\; \rho = \sum_{i=1}^{n} t_i \rho_i \text{ with } t_i \geq 0 \text{ and } \sum_{i=1}^{n} t_i = 1 \right\}$$

i.e. the entanglement entropy is defined to be the infimum over all decompositions of ρ as a convex combination of pure states.

This measure of entanglement is also called the "entanglement of formation". (Note: There is an equivalent operational definition...)

For general density matrices $\rho \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$, one has that

- $S_E(\rho) = 0 \iff \rho$ is separable
- $0 \leq S_E(\rho) \leq \min\big(\ln(\dim(\mathcal{H}_1)), \ln(\dim(\mathcal{H}_2)) \big)$

Thus S_E is a measure of entanglement and any density matrix ρ for which $S_E(\rho)$ is maximal is said to correspond to a maximally entangled state.

Ex Let $H_1 = \mathbb{C}^2$ and $H_2 = \mathbb{C}^2$.

The maximally entangled states in this context are called Bell States. (The name comes from John Bell who wrote a famous paper on them in 1964.)

All Bell states can be described as follows:

Let $\{e_1, e_2\} \subset H_1$ and $\{f_1, f_2\} \subset H_2$ be orthonormal bases in $\mathbb{C}^2$.

Any state of the form

$$\frac{e_1 \otimes f_1 + e_2 \otimes f_2}{\sqrt{2}}$$ is called a Bell state.

The pair of vectors in a Bell state ($e_1 \otimes f_1$ and $e_2 \otimes f_2$ in the example above) are often called EPR pairs.

This label comes from a famous paper in 1935 by Einstein, Podolsky, and Rosen where they questioned some fundamentals of quantum mechanics

Let us check that a specific Bell state is maximally entangled.

Let $\{|0\rangle, |1\rangle\}$ denote an ONB of $\mathbb{C}^2$.

<u>Take</u> $\psi = \frac{1}{\sqrt{2}} \left(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle \right)$

Let $\rho = |\psi\rangle\langle\psi|$.

Consider $\rho_1 = \mathrm{Tr}_{H_2}[\rho]$ and calculate:

$$\langle i, \rho_1 j \rangle = \sum_{k=1}^{2} \langle i \otimes k, |\psi\rangle\langle\psi| \, j \otimes k \rangle$$

$$= \begin{cases} 0 & i \neq j \\ \left(\frac{1}{\sqrt{2}}\right)^2 & i = j \end{cases}$$

$\Rightarrow \rho_1 = \frac{1}{2} \mathbb{1}$

$\Rightarrow \rho_1$ is maximally mixing

$\Rightarrow S(\rho_1) = \ln(2) = \min(\ln(2), \ln(2))$ ✓

We now prove that: If $\rho \in B(H_1 \otimes H_2)$ is the density matrix associated to a pure state, then

$$S(\rho_1) = S(\rho_2)$$

where $\rho_1 = \mathrm{Tr}_{H_2}[\rho]$, $\rho_2 = \mathrm{Tr}_{H_1}[\rho]$, and $S(\rho)$ is the von-Neumann entropy.

First, an important result from linear algebra.

<u>Theorem</u>: (Schmidt Decomposition)

Let H_1 and H_2 be finite dimensional (non-zero) complex Hilbert spaces. Let $\psi \in H_1 \otimes H_2$ with $\|\psi\| = 1$.

Then, there are orthonormal bases $\{e_j\}_{j \geq 1}$ and $\{f_k\}_{k \geq 1}$ of H_1 and H_2 respectively and a unique sequence of numbers $s_1 \geq s_2 \geq \cdots \geq 0$ with $\sum_{j \geq 1} s_j^2 = 1$ for which

$$\psi = \sum_{j \geq 1} s_j \, e_j \otimes f_j . \tag{$*$}$$

In particular, ψ is a product vector on $H_1 \otimes H_2$ if and only if $s_1 \neq 0$ and $s_j = 0$ for all $j \geq 2$.

Proof of Schmidt Decomposition

Note that if $(*)$ holds for ONBs $\{e_j\}$ and $\{f_k\}$, then

$$1 = \|\psi\|^2 = \sum_{j\geq 1} s_j^2 \qquad \text{since } \{e_j \otimes f_k\} \text{ is an ONB of } H_1 \otimes H_2.$$

Now, let $\{\tilde{e}_j\}_{j=1}^m$ and $\{\tilde{f}_k\}_{k=1}^n$ be ONBs of H_1 and H_2 respectively. Since $\psi \in H_1 \otimes H_2$, it is clear that

$$\psi = \sum_{j,k\geq 1} \psi_{jk}\, \tilde{e}_j \otimes \tilde{f}_k \qquad \text{as } \{\tilde{e}_j \otimes \tilde{f}_k\} \text{ is an ONB of } H_1 \otimes H_2.$$

Denote by $A \in \mathbb{C}^{m\times n}$ the matrix with entries $a_{jk} = \psi_{jk}$.

The matrix A has a singular value decomposition:

$$A = U D V^* \qquad \text{where } U \in M_m \text{ and } V \in M_n \text{ are unitary}$$

and $D \in \mathbb{C}^{m\times n}$ is diagonal with non-negative diagonal entries corresponding to the singular values of A. These singular values are uniquely determined by A which is uniquely determined by ψ and the basis $\{\tilde{e}_j \otimes \tilde{f}_k\}$. (They are, in fact, independent of the initial choice of basis...)

Recall that

$$\psi_{jk} = A_{jk} = (UDV^*)_{jk}$$

$$= \sum_{\ell} U_{j\ell} (DV^*)_{\ell k}$$

$$= \sum_{\ell,\ell'} U_{j\ell} D_{\ell\ell'} V^*_{\ell' k} \quad \leftarrow \text{D is diagonal}$$

$$= \sum_{\ell} \sigma_\ell(A) U_{j\ell} V^*_{\ell,k}$$

$$\Rightarrow \psi = \sum_{j,k\geq 1} \psi_{jk} \tilde{e}_j \otimes \tilde{f}_k = \sum_{j,k\geq 1} \left(\sum_{\ell} \sigma_\ell(A) U_{j\ell} V^*_{\ell,k} \right) \tilde{e}_j \otimes \tilde{f}_k$$

$$= \sum_{\ell} \sigma_\ell(A)\, e_\ell \otimes f_\ell$$

where $e_\ell = \sum_{j\geq 1} U_{j\ell} \tilde{e}_j$ and $f_\ell = \sum_{k\geq 1} V^*_{\ell,k} \tilde{f}_k$

As U and V were unitary, one checks that $\{e_\ell\}$ and $\{f_\ell\}$ are both orthonormal sets in H_1 and H_2 respectively.

(They may be extended to a full ONB if necessary.)

This completes the proof.

Now, let $\rho \in B(H_1 \otimes H_2)$ be a density matrix associated to a pure state. Since ρ is pure, $\exists\, \psi \in H_1 \otimes H_2$ with $1 = \|\psi\|$ and $\rho = |\psi\rangle\langle\psi|$.

Let $\{e_j\}$ and $\{f_k\}$ denote the ONBs of H_1 and H_2 whose existence is guaranteed by the Schmidt decomposition of ψ, i.e. write

$$\psi = \sum_{j \geq 1} s_j\, e_j \otimes f_j$$

use that $s_k \geq 0$

$$\Rightarrow \rho = |\psi\rangle\langle\psi| = \sum_{j,k \geq 1} s_j s_k\, |e_j \otimes f_j\rangle\langle e_k \otimes f_k|$$

Clearly $\rho_1 = \mathrm{Tr}_{H_2}[\rho] \in B(H_1)$. Calculate the matrix entries of ρ_1 in the basis $e = \{e_j\}$.

$$\langle e_i, \rho_1 e_j\rangle = \sum_{k \geq 1} \langle e_i \otimes f_k, \rho\, e_j \otimes f_k\rangle$$

$$= \sum_{k \geq 1} \sum_{\ell, \ell' \geq 1} s_\ell s_{\ell'} \langle e_i \otimes f_k, e_\ell \otimes f_\ell\rangle \langle e_{\ell'} \otimes f_{\ell'}, e_j \otimes f_k\rangle$$

$$= \sum_{k \geq 1} \sum_{\ell, \ell' \geq 1} s_\ell s_{\ell'}\, \delta_{i,\ell}\, \delta_{k,\ell} \cdot \delta_{\ell',j} \cdot \delta_{\ell',k}$$

$$= \sum_{k \geq 1} s_i \delta_{ki} s_j \delta_{jk} = \begin{cases} 0 & \text{if } j \neq i \\ s_i^2 & \text{if } i = j \end{cases}$$

Thus the partial trace ρ_1 is diagonal in this basis

i.e. $\rho_1 = \sum_i s_i^2 |e_i\rangle\langle e_i|$ and thus

$$\Rightarrow \quad S(\rho_1) = - \sum_i s_i^2 \ln(s_i^2)$$

A similar calculation shows that

$$S(\rho_2) = - \sum_i s_i^2 \ln(s_i^2)$$

This proves the claim.