Lecture 13

Last class we defined C^*-algebras.
This class we will investigate maps between C^*-algebras and states on C^*-algebras.

On maps between C^*-algebras

Linear maps between vector spaces that "preserve structures" are generally called morphisms. In the context of C^* algebras, these are typically $*$-morphisms.

Definition: Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras. A map $\pi : \mathcal{A} \to \mathcal{B}$ is called a $*$-morphism if

i) $\pi(\alpha A + \beta B) = \alpha \pi(A) + \beta \pi(B)$ (linear)

for all $A, B \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$.

ii) $\pi(AB) = \pi(A) \cdot \pi(B)$ (π preserves products)

for all $A, B \in \mathcal{A}$.

iii) $\pi(A^*) = \pi(A)^*$ (π preserves the involution)

for all $A \in \mathcal{A}$

Note: In general, the phrase algebra morphism refers to a map satisfying properties i) and ii) above. Property iii), makes π into a $*$-morphism.

We will only consider $*$-morphisms.

- A $*$-morphism $\pi : \mathcal{A} \to \mathcal{B}$ is said to be unit preserving if

$$\pi(\mathbb{1}_{\mathcal{A}}) = \mathbb{1}_{\mathcal{B}}.$$

- Any $*$-morphism $\pi : \mathcal{A} \to \mathcal{B}$ is "positivity preserving".

In fact, let $A \in \mathcal{A}$ be positive. Then there is some $B \in \mathcal{A}$ for which $A = B^* B$. Note then that

$$\pi(A) = \pi(B^* B) \overset{ii)}{=} \pi(B^*)\pi(B) \overset{iii)}{=} \pi(B)^* \pi(B)$$

and so $\pi(A) \in \mathcal{B}$ is positive.

Recall that if $\mathcal{X}$ and $\mathcal{Y}$ are normed spaces and $T : \mathcal{X} \to \mathcal{Y}$ is a linear map, then T is said to be bounded if there is a number $C \geq 0$ for which:

$$\|Tx\|_{\mathcal{Y}} \leq C \cdot \|x\|_{\mathcal{X}} \quad \text{for all } x \in \mathcal{X}.$$

The norm of T is then defined to be the quantity:

$$\|T\| := \sup_{x \in \mathcal{X}:} \frac{\|Tx\|}{\|x\|}$$

This quantity is a norm on the bounded linear maps from $\mathfrak{X}$ to $\mathfrak{Y}$.

Here is a useful fact.

Proposition Let $\mathfrak{A}$ and $\mathfrak{B}$ be unital C^*-algebras and $\pi : \mathfrak{A} \to \mathfrak{B}$ be a $*$-morphism. Then

$$\|\pi(A)\| \leq \|A\| \qquad \text{for all } A \in \mathfrak{A}.$$

In this case, π is a bounded linear map between $\mathfrak{A}$ and $\mathfrak{B}$ and $\|\pi\| \leq 1$.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be unital C^*-algebras.
A $*$-morphism π from $\mathfrak{A}$ to $\mathfrak{B}$ is said to be a $*$-isomorphism if it is one to one and onto.

Recall: • one to one means injective, i.e.

$$\ker(\pi) = \{A \in \mathfrak{A} : \pi(A) = 0\} = \{0\}.$$

• onto means surjective, i.e.

$$\pi(\mathfrak{A}) = \{B \in \mathfrak{B} : B = \pi(A) \text{ for some } A \in \mathfrak{A}\} = \mathfrak{B}.$$

Note: For any $*$-morphism, $\pi(\mathfrak{A}) \subset \mathfrak{B}$ is a self-adjoint subset.

On Representations

<u>Definition</u> Let $\mathfrak{A}$ be a C^*-algebra. A <u>representation</u> of $\mathfrak{A}$ is a pair $(\mathcal{H}, \pi)$ where $\mathcal{H}$ is a complex Hilbert space and π is a $*$-morphism from $\mathfrak{A}$ to $B(\mathcal{H})$.
The representation $(\mathcal{H}, \pi)$ of $\mathfrak{A}$ is said to be <u>faithful</u> if and only if π is a $*$-isomorphism from $\mathfrak{A}$ to $\pi(\mathfrak{A})$.

<u>Note</u>: If $\mathfrak{A}$ is a C^*-algebra and $(\mathcal{H}, \pi)$ is a representation of $\mathfrak{A}$, then

- $\mathcal{H}$ is called the representation space
- the elements $\pi(A) \in B(\mathcal{H})$ are called the representatives of $\mathfrak{A}$
- π is often called a representation of $\mathfrak{A}$ on $\mathcal{H}$.

<u>Proposition</u> Let $\mathfrak{A}$ be a C^* algebra and $(\mathcal{H}, \pi)$ a representation of $\mathfrak{A}$. The representation is faithful if and only if it satisfies one (and then all) of the following equivalent conditions:

i) $\operatorname{Ker}(\pi) = \{0\}$.

ii) $\|\pi(A)\| = \|A\|$ for all $A \in \mathfrak{A}$

iii) for all $A \in \mathfrak{A}$ with $A \geq 0$ and $A \neq 0$,

$$\pi(A) \geq 0 \quad \text{and} \quad \pi(A) \neq 0.$$

<u>Definition</u> Let $\mathfrak{A}$ be a C^*-algebra.
A $*$-isomorphism π from $\mathfrak{A}$ to $\mathfrak{A}$ is called an <u>automorphism</u> of $\mathfrak{A}$.

In quantum mechanics, the Heisenberg dynamics will be an automorphism of the observable algebra.

<u>Corollary</u>: Let $\mathfrak{A}$ be a C^*-algebra and π an automorphism on $\mathfrak{A}$. Then

$$\|\pi(A)\| = \|A\| \quad \text{for all } A \in \mathfrak{A}.$$

In words, an automorphism π preserves norm.

<u>States on a C^*-algebra</u>

Let $\mathfrak{A}$ be a unital C^*-algebra. By $\mathfrak{A}^*$ we denote the <u>dual</u> of $\mathfrak{A}$, i.e. $\mathfrak{A}^*$ is the collection of bounded linear maps $f: \mathfrak{A} \to \mathbb{C}$. For any $f \in \mathfrak{A}^*$, we denote by

$$\|f\| = \sup \{ |f(A)| : A \in \mathfrak{A} \text{ and } \|A\| = 1 \}$$

the norm of f. This quantity is a norm on $\mathfrak{A}^*$.

Definition Let $\mathcal{A}$ be a unital C^*-algebra.

- $\omega \in \mathcal{A}^*$ is said to be positive if and only if

$$\omega(A^*A) \geq 0 \quad \text{for all } A \in \mathcal{A}.$$

- $\omega \in \mathcal{A}^*$ is said to be a state if and only if

ω is positive and $\|\omega\| = 1$.

By definition, positive linear functionals on $\mathcal{A}$ take positive elements to non-negative real numbers.

Note also that:

If $A, B \in \mathcal{A}_{sa}$ and $A \geq B$, then $A - B \geq 0$ i.e.

$A - B = C^*C$ for some $C \in \mathcal{A}$. In this case,

$$0 \leq \omega(C^*C) = \omega(A-B) = \omega(A) - \omega(B).$$

for any positive linear functional on $\mathcal{A}$.

I.E. If $A, B \in \mathcal{A}_{sa}$ and $A \geq B$, then $\omega(A) \geq \omega(B)$ for all positive linear functionals ω on $\mathcal{A}$.

In particular, this is true for all states on $\mathcal{A}$.

There are important connections between states on $\mathcal{A}$ and representations on $\mathcal{A}$.

Let $\mathcal{A}$ be a unital C^*-algebra.

Let $(\mathcal{H}, \pi)$ be a representation of $\mathcal{A}$.

For any non-zero $\psi \in \mathcal{H}$, define

$$\omega_\psi(A) = \langle \psi, \pi(A)\psi \rangle \quad \text{for all } A \in \mathcal{A}.$$

It is clear that ω_ψ is a linear functional on $\mathcal{A}$, i.e. $\omega_\psi \in \mathcal{A}^*$.

It is also clear that ω_ψ is positive:

$$\omega_\psi(A^*A) = \langle \psi, \pi(A^*A)\psi \rangle = \langle \psi, \pi(A)^* \pi(A)\psi \rangle$$

$$= \langle \pi(A)\psi, \pi(A)\psi \rangle$$

$$= \|\pi(A)\psi\|^2 \geq 0.$$

In this case, if $\|\psi\| = 1$ and π is non-degenerate (i.e. $\pi(A) \not\equiv 0$), then one can check that $\|\omega_\psi\| = 1$ and hence ω_ψ is a state on $\mathcal{A}$. States of this type are called vector states of the representation $(\mathcal{H}, \pi)$. One can prove that every state on a C^*-algebra $\mathcal{A}$ is a vector state in a suitable representation.

General Properties of States on C^*-algebras

Let us collect some basic facts concerning states on a C^*-algebra

<u>Lemma</u> (Cauchy-Schwarz) Let $\mathcal{A}$ be a C^*-algebra and ω a positive linear functional on $\mathcal{A}$. Then

i) $\omega(A^*B) = \overline{\omega(B^*A)}$ for all $A, B \in \mathcal{A}$.

ii) $|\omega(A^*B)|^2 \leq \omega(A^*A)\,\omega(B^*B)$ for all $A, B \in \mathcal{A}$.

<u>proof:</u>

Let $A, B \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. Then by positivity of ω

$$0 \leq \omega((\lambda A + B)^*(\lambda A + B))$$

$$= |\lambda|^2 \omega(A^*A) + \bar{\lambda}\,\omega(A^*B) + \lambda\,\omega(B^*A) + \omega(B^*B)$$

The final equality using linearity of ω.

We conclude that

$$\mathrm{Im}[\bar{\lambda}\,\omega(A^*B) + \lambda\,\omega(B^*A)] = 0 \qquad \text{for all } \lambda \in \mathbb{C}.$$

$$\left.\begin{array}{l} \cdot\ \lambda = i \Rightarrow \mathrm{Re}[\omega(A^*B)] = \mathrm{Re}[\omega(B^*A)] \\ \cdot\ \lambda = 1 \Rightarrow \mathrm{Im}[\omega(A^*B)] = -\mathrm{Im}[\omega(B^*A)] \end{array}\right\} \Rightarrow \text{i)} \checkmark$$

To see ii), take $\lambda = t\omega(A^*B)$ for $t \in \mathbb{R}$.

The quadratic polynomial:

$$p(t) = |t\omega(A^*B)|^2 \omega(A^*A) + t|\omega(A^*B)|^2 + t|\omega(A^*B)|^2 + t\omega(B^*B)$$

$$= at^2 + bt + c$$

with $a = |\omega(A^*B)|^2 \omega(A^*A)$, $b = 2|\omega(A^*B)|^2$, and $c = \omega(B^*B)$

is non-negative. Thus

$$b^2 - 4ac \leq 0 \Rightarrow 4|\omega(A^*B)|^4 - 4|\omega(A^*B)|^2 \omega(A^*A)\omega(B^*B) \leq 0$$

$$\Rightarrow 4|\omega(A^*B)|^2 \left(|\omega(A^*B)|^2 - \omega(A^*A)\omega(B^*B) \right) \leq 0$$

$\Rightarrow$ either $|\omega(A^*B)|^2 = 0$ ✓

or ii) is true ✓

Here is another important result.

<u>Corollary</u> Let $\mathfrak{A}$ be a non-trivial unital C^*-algebra and ω a positive linear functional on $\mathfrak{A}$.

i) $\omega(A^*) = \overline{\omega(A)}$ for all $A \in \mathfrak{A}$.

ii) $\omega(\mathbb{1}) = \|\omega\| = \sup\{\omega(A^*A) : \|A\| = 1\}$

iii) $|\omega(A)|^2 \leq \omega(A^*A)\|\omega\|$ for all $A \in \mathfrak{A}$.

iv) $|\omega(A^*BA)| \leq \omega(A^*A)\|B\|$ for all $A, B \in \mathfrak{A}$.

<u>Proof:</u>

i) This follows from the above Lemma i) with $B = \mathbb{1}$.

ii) The 1st equality follows from two inequalities. First,

$$0 \leq \omega(\mathbb{1}^*\mathbb{1}) = \omega(\mathbb{1}) = \frac{\omega(\mathbb{1})}{\|\mathbb{1}\|} \leq \|\omega\|$$

here we have used that $\mathbb{1}^*\mathbb{1} = \mathbb{1}$ and $\|\mathbb{1}\| = 1$.

Next, for any $A \in \mathfrak{A}$,

$$(*) \quad |\omega(A)|^2 = |\omega(\mathbb{1}^*A)|^2 \overset{ii)}{\leq} \omega(\mathbb{1}^*\mathbb{1})\omega(A^*A) = \omega(\mathbb{1}) \cdot \omega(A^*A)$$

If $A \in \mathcal{A}$ has $\|A\| = 1$, then. $B = A^*A$

satisfies $\|B\| = \|A^*A\| = \|A\|^2 = 1$.

$\Rightarrow$ ($*$) implies

$$|\omega(A)|^2 \leq \omega(\mathbb{1}) \cdot \omega(B) \leq \omega(\mathbb{1}) \cdot \|\omega\|$$

$$\Rightarrow \quad \|\omega\|^2 \leq \omega(\mathbb{1}) \cdot \|\omega\|$$

Now, if $\|\omega\| = 0$, then $\omega(\mathbb{1}) = 0$ by above.

otherwise $\omega(\mathbb{1}) = \|\omega\|$. ✓

The inequality in iii) now follows from ($*$).
The 2nd equality in ii) now follows from iii).

To prove iv). Fix $A \in \mathcal{A}$. Define $\omega_A : \mathcal{A} \to \mathbb{C}$

by setting

$$\omega_A(B) = \omega(A^* B A)$$

Clearly ω_A is linear.
Also $\omega_A(B^*B) = \omega(A^* B^* B A) = \omega((BA)^* BA) \geq 0$

$\Rightarrow$ ω_A is positive.

If $B = 0$, ✓ If $B \neq 0$, then.

$$\frac{|\omega(A^* B A)|}{\|B\|} = \left|\omega\left(A^* \frac{B}{\|B\|} A\right)\right| = \left|\omega_A\left(\frac{B}{\|B\|}\right)\right| \leq \|\omega_A\| = \omega_A(\mathbb{1})$$
$$= \omega(A^*A) \quad ✓$$

By part (iv) above, we have that:

Let $A \in \mathcal{A}$ with $\omega(A^*A) \neq 0$.

Then

$$\widetilde{\omega}_A(B) = \frac{\omega(A^*BA)}{\omega(A^*A)} \quad \text{for all } B \in \mathcal{A}$$

defines a state on $\mathcal{A}$.

This is the quantum mechanical analogue of:

Take a function $f : [0,1] \to (0,\infty)$. for which

$$\int_0^1 f(x)\,dx < \infty.$$

Define a normalized measure μ on $[0,1]$ by setting.

$$\int_0^1 g(x)\,d\mu(x) = \frac{\int_0^1 g(x) f(x)\,dx}{\int_0^1 f(x)\,dx}$$

i.e. μ is the measure with density

$$\rho(x) = \frac{f(x)}{\int_0^1 f(x)\,dx}.$$