On Calculus in a Banach Space

Let $\mathbb{X}$ be a Banach space, i.e. $\mathbb{X}$ is a complete normed space.
For the results considered here, $\mathbb{X}$ can be a vectorspace over $\mathbb{R}$ or $\mathbb{C}$.

Let $I \subset \mathbb{R}$ be an open interval.

<u>Definition</u> A map $f: I \to \mathbb{X}$ is said to be continuous at $t_0 \in I$ if

$$\lim_{t \to t_0} \| f(t) - f(t_0) \| = 0.$$

If f is continuous at all $t \in I$, then f is said to be continuous on I (or just continuous). We often use $C(I, \mathbb{X})$ to denote the collection of all such continuous functions on I.

<u>Note</u>: If $I = [a,b] \subset \mathbb{R}$ is a compact interval, then "one-sided" notions of continuity are defined at the end points.

- $f: [a,b] \to \mathbb{X}$ is continuous at a if

$$\lim_{t \to a^+} \| f(t) - f(a) \| = 0.$$

- $f: [a,b] \to \mathbb{X}$ is continuous at b if

$$\lim_{t \to b^-} \| f(t) - f(b) \| = 0.$$

Proposition If $f : I \to X$ is continuous at $t_0 \in I$, then $g : I \to [0,\infty)$ defined by setting

$$g(t) = \|f(t)\| \quad \text{for all } t \in I$$

is continuous at $t_0 \in I$.

Proof:

Any normed space satisfies the triangle inequality:

$$\|x+y\| \le \|x\| + \|y\| \quad \text{for all } x, y \in X.$$

A consequence of this is the "reverse triangle inequality":

$$\big| \|x\| - \|y\| \big| \le \|x-y\| \quad \text{for all } x, y \in X.$$

Check this!

In this case, setting $x = f(t)$ and $y = f(t_0)$ we see that

$$|g(t) - g(t_0)| = \big| \|x\| - \|y\| \big| \le \|x-y\| = \|f(t) - f(t_0)\|$$

and so continuity of g follows from continuity of f.

Two immediate facts follow:

- If $f: [a,b] \to X$ is continuous, then f is bounded, i.e.

$$\sup_{a \le t \le b} \| f(t) \| < \infty .$$

- If $f: [a,b] \to X$ is continuous, then f is uniformly continuous, i.e. for every $\varepsilon > 0$, there is a $\delta > 0$ for which

$$\| f(x) - f(y) \| \le \varepsilon \quad \text{whenever} \quad |x - y| < \delta \qquad (\text{Here } x, y \in [a,b].)$$

<u>Definition</u> A map $f: I \to X$ is said to be differentiable at $t_0 \in I$ if

$$\lim_{h \to 0} \frac{f(t_0 + h) - f(t_0)}{h} \quad \text{exists in } X.$$

Two common notations for this limit are: $f'(t_0)$ and $\frac{d}{dt} f(t_0)$.

If f is differentiable at all $t \in I$, then f is said to be differentiable on I (or just differentiable).

<u>Note</u>: If $I = [a,b]$, then differentiability at endpoints is defined similarly to continuity; one-sided derivatives...

It is clear that if $f, g : I \to \mathfrak{X}$ are both continuous (differentiable) at t_0, then for any scalars α and β

$$\alpha f + \beta g \quad \text{is continuous (differentiable) at } t_0 \text{ as well.}$$

The following is a Homework problem.
It considers the case of $\mathfrak{X} = B(\mathcal{H})$, but analogues hold for general Banach algebras...

<u>HW</u> Let $\mathcal{H}$ be a complex Hilbert space and $I \subset \mathbb{R}$ an open interval.

i) Let $A, B : I \to B(\mathcal{H})$ be continuous maps.
Show that $C : I \times I \to B(\mathcal{H})$ given by

$$C(s,t) = A(s)B(t) \qquad \text{for all } s,t \in I$$

is jointly continuous, i.e. for any $(s_0, t_0) \in I \times I$,

$$\lim_{(s,t) \to (s_0,t_0)} \| C(s,t) - C(s_0,t_0) \| = 0.$$

ii) Let $A, B : I \to B(\mathcal{H})$ be differentiable maps.
Show that $C : I \times I \to B(\mathcal{H})$ given by

$$C(s,t) = A(s)B(t) \qquad \text{for all } s,t \in I$$

is separately differentiable.

i.e. the map $t \mapsto C(s,t)$ is differentiable for each fixed $s \in I$

and the map $s \mapsto C(s,t)$ is differentiable for each fixed $t \in I$.

iii) Let $A, B : I \to B(H)$ be differentiable maps.

Show that $C : I \to B(H)$ given by

$$C(t) = A(t)B(t) \qquad \text{for all } t \in I$$

is differentiable. Find $C'(t)$.

<u>Example</u>: Let X be a Banach space and $I \subset \mathbb{R}$ be an open interval. Let $f : I \to \mathbb{C}$ be a differentiable function. For any $x \in X$, define $g : I \to X$ by setting

$$g(t) = f(t)\,x \qquad \text{for all } t \in I.$$

Show that $g'(t) = f'(t)x$ for all $t \in I$.

Use this to conclude the following:

Let H be finite dimensional. Let $H^* = H \in B(H)$.

Prove that:

$$\frac{d}{dt} e^{-itH} = -iHe^{-itH} = -ie^{-itH}H.$$

On Integration

We now introduce integration for certain Banach space valued functions. To keep things simple, we restrict our attention to the Riemann integral. With measure theory, one could introduce the Bochner integral ..., but we will not need that for our quantum spin systems.

Let $\mathbb{X}$ be a Banach space.

Keep in mind our two most important examples:

- $\mathbb{X} = \mathcal{H}$ a Hilbert space.

For each $x \in \mathbb{X}$, $\|x\| = \sqrt{\langle x, x \rangle}$.

- $\mathbb{X} = \mathcal{B}(\mathcal{H})$ for $\mathcal{H}$ a Hilbert space.

For each $A \in \mathbb{X}$, $\|A\| = \sup\limits_{\substack{\psi \in \mathcal{H}: \\ \psi \neq 0}} \dfrac{\|A\psi\|}{\|\psi\|}$.

Again, for this integration theory, $\mathbb{X}$ can be a vector space over $\mathbb{R}$ or $\mathbb{C}$.

Let $I = [a,b] \subset \mathbb{R}$ be a compact interval.

Consider the set of bounded functions from I to X, i.e.

$$\mathcal{F}([a,b], X) := \{ f : [a,b] \to X \mid \|f\|_\infty = \sup_{a \le t \le b} \|f(t)\| < \infty \}.$$

It is clear that, with respect to the usual arithmetic for functions, $\mathcal{F}([a,b], X)$ is a vector space.

H.W. Show that $\|\cdot\|_\infty$ is a norm on $\mathcal{F}([a,b], X)$ and moreover, show that $\mathcal{F}([a,b], X)$ is a Banach space with respect to this norm.

We begin by considering a class of "simple" functions for which there is a "natural" notion of integral.

Let $\mathcal{S}([a,b], X) = \{ f : [a,b] \to X \mid f \text{ is a "step function"} \}.$

More precisely, f is a step-function if and only if:

- there is an integer $n \ge 1$ and a partition $\{t_j\}_{j=0}^{n}$ of $[a,b]$, i.e.

$$a = t_0 < t_1 < t_2 < \cdots < t_n = b$$

- there is a collection $\{x_j\}_{j=0}^{n-1}$ of n points $x_j \in X$

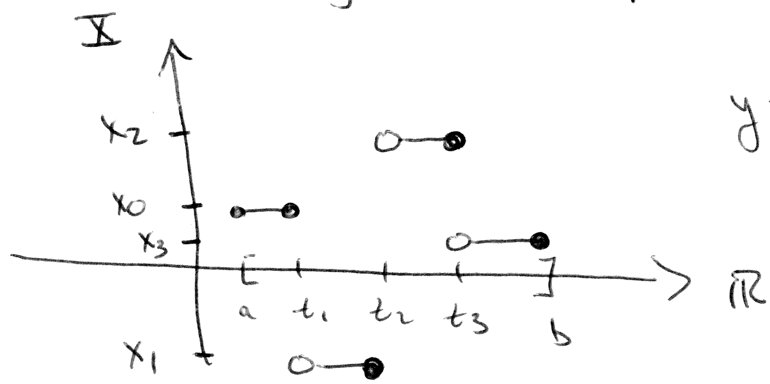
and

$$(*)\quad f(t) = x_0 \chi_{[a,t_1]}(t) + \sum_{j=1}^{n-1} x_j \chi_{(t_j,t_{j+1}]}(t)$$

Here for any set $A \subset \mathbb{R}$, $\chi_A : \mathbb{R} \to \mathbb{R}$ is the characteristic function of the set A, i.e.

$$\chi_A(t) = \begin{cases} 1 & \text{if } t \in A \\ 0 & \text{if } t \in \mathbb{R}/A. \end{cases}$$

In principle, one can graph a step-function:



$$y = f(t) = x_0 \chi_{[a,t_1]}(t) + \sum_{j=1}^{3} x_j \chi_{(t_j,t_{j+1}]}(t).$$

It is clear that

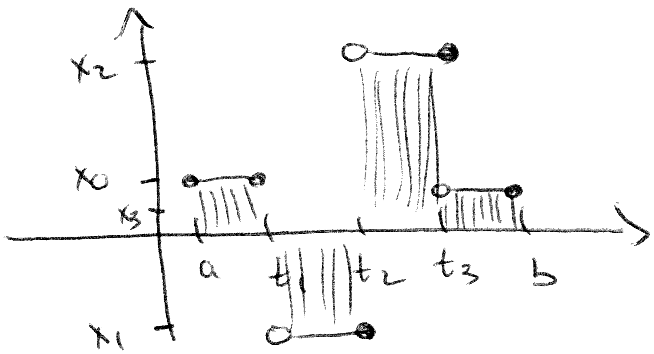
$$\mathcal{A}([a,b],\mathbb{X}) \subset \mathcal{F}([a,b],\mathbb{X}).$$

In fact, if $f \in \mathcal{A}([a,b],\mathbb{X})$ has the form $(*)$ then

$$\|f\|_\infty = \max\{\|x_j\| : 0 \le j \le n-1\}.$$

One also readily checks that $\mathcal{A}([a,b],\mathbb{X})$ is a subspace of $\mathcal{F}([a,b],\mathbb{X})$.

Based on our experience in calculus, it is clear what the integral of a "step-function" should be:



integral of f should be sums of these "areas".

Define a map $I : \mathcal{A}([a,b], \mathbb{X}) \to \mathbb{X}$ by setting

$$I(f) = \sum_{j=0}^{n-1} (t_{j+1} - t_j) x_j$$

for any $f \in \mathcal{A}([a,b], \mathbb{X})$ having the form $(*)$.

<u>HoWo</u> Check that $I : \mathcal{A}([a,b], \mathbb{X}) \to \mathbb{X}$ is a well-defined linear map. (Here well-defined means independent of the particular representation of the step-function ...)

We refer to $I(f) \in \mathbb{X}$ as the <u>integral</u> of this step function f.

The following Proposition describes a larger class of functions for which this Riemann integral exists.

To ease notation, regard $\mathbb{X}$ and $[a,b]$ as fixed.

Set $\mathcal{A} = \mathcal{A}([a,b], \mathbb{X})$.

Denote by $\overline{\mathcal{A}}$ the norm closure of the subspace $\mathcal{A}([a,b], \mathbb{X})$

$\subset \mathcal{F}([a,b], \mathbb{X})$.

<u>Proposition</u> (Riemann Integral)

Let $\mathbb{X}$ be a Banach space and $[a,b] \subset \mathbb{R}$ an interval.

i) The map $I : \mathcal{A} \to \mathbb{X}$ satisfies

$$\| I(f) \| \leq (b-a) \|f\|_\infty \quad \text{for all } f \in \mathcal{A}.$$

ii) There is a unique linear map $\overline{I} : \overline{\mathcal{A}} \to \mathbb{X}$ for which

- $\overline{I}(f) = I(f)$ for all $f \in \mathcal{A}$
- $\| \overline{I}(f) \| \leq (b-a) \cdot \|f\|_\infty$.

For any $f \in \overline{\mathcal{A}}$, $\overline{I}(f)$ is called the Riemann integral of f.

iii) $C([a,b], \mathbb{X}) \subset \overline{\mathcal{A}}$.

Proof:

i) Let $f \in A$ have the form $(*)$.
Then by definition

$$I(f) = \sum_{j=0}^{n-1} (t_{j+1} - t_j) x_j$$

$$\Rightarrow \| I(f) \| \leq \sum_{j=0}^{n-1} (t_{j+1} - t_j) \| x_j \| \leq \max_{0 \leq j \leq n-1} \| x_j \| \cdot \sum_{j=0}^{n-1} (t_{j+1} - t_j)$$

$$= \| f \|_\infty \cdot (b-a)$$

ii) Since A is a subspace of a normed space, we can apply the bounded linear transformation theorem to I (Here we use the bound proven in i).) This is the claim in ii).

iii) Let $f \in C([a,b], X)$. As previously discussed, any such f is uniformly continuous. Thus for any $\varepsilon > 0$, there is a $\delta > 0$ for which:

$$\| f(x) - f(y) \| < \varepsilon \quad \text{whenever} \quad |x - y| < \delta.$$

In this case, for any partition $\{t_j\}_{j=0}^{n}$ of $[a,b]$ for which

$$\operatorname{mesh}(\{t_j\}_{j=0}^{n}) = \max\{ |t_{j+1} - t_j| : 0 \leq j \leq n-1 \} < \delta$$

and any choice of values

$$t_j^* \in [t_j, t_{j+1}] \quad \text{for } 0 \le j \le n-1$$

The function $f_n : [a,b] \to X$ given by

$$f_n(t) = f(t_0^*)\,\chi_{[a,t_1]}(t) + \sum_{j=1}^{n-1} f(t_j^*)\,\chi_{(t_j, t_{j+1}]}(t)$$

satisfies:

- $f_n \in A$

and

- $\| f - f_n \|_\infty \le \varepsilon$

Thus $f \in \overline{A}$ and so $C([a,b], X) \subset \overline{A}$ as claimed.

Note: a consequence of this result is another fact from calculus.

If $f \in C([a,b], X)$ then for a sequence as above

$$\overline{I}(f) = \lim_{n\to\infty} \overline{I}(f_n) = \lim_{n\to\infty} I(f_n)$$

$$= \lim_{n\to\infty} \sum_{j=0}^{n-1} (t_{j+1} - t_j)\, f(t_j^*)$$

In words, the Riemann integral is the limit of Riemann sums. Here the mesh size must go to zero, but the value is independent of the points at which the function is evaluated (on the subintervals).

Some notation

Let $a \le \alpha < \beta \le b$ and take $f \in \overline{\mathcal{A}}([a,b], \mathbb{X})$.

Let $\{f_n\}_{n \ge 1}$ be a sequence with $f_n \in \mathcal{A}([a,b], \mathbb{X})$ for all $n \ge 1$ satisfying.

$$\|f_n - f\|_\infty \to 0 \quad \text{as} \quad n \to \infty.$$

For each $n \ge 1$, set $g_n = \chi_{[\alpha,\beta]} \cdot f_n \in \mathcal{A}([a,b], \mathbb{X})$.

It is clear that

$$\|g_n - g\|_\infty \to 0 \quad \text{as} \quad n \to \infty$$

where $g = \chi_{[\alpha,\beta]} \cdot f$.

In this case, we conclude that for each $f \in \overline{\mathcal{A}}([a,b], \mathbb{X})$ the function $\chi_{[\alpha,\beta]} \cdot f \in \overline{\mathcal{A}}([a,b], \mathbb{X})$ for all α, β as above.

The notation

$$\int_\alpha^\beta f(t)\,dt = \overline{I}(\chi_{[\alpha,\beta]} f)$$

is often used. Let us further define:

$$\int_\beta^\alpha f(t)\,dt := -\overline{I}(\chi_{[\alpha,\beta]} f) = -\int_\alpha^\beta f(t)\,dt$$

The following lemma summarizes many properties of this Riemann integral. Most are familiar from calculus, and the proof of these statements is a homework problem.

<u>Lemma</u>: Let $\mathbb{X}$ be a Banach space, $[a,b] \subset \mathbb{R}$ be a compact interval, and $f \in \mathcal{A}([a,b], \mathbb{X})$.

i) Let $a \le \alpha < \beta \le b$. One has that

$$\left\| \int_\alpha^\beta f(t)\,dt \right\| \le (\beta - \alpha) \cdot \sup \{ \|f(t)\| : \alpha \le t \le \beta \}.$$

ii) For any $\alpha, \beta, \gamma \in [a,b]$, one has that

$$\int_\alpha^\beta f(t)\,dt = \int_\alpha^\gamma f(t)\,dt + \int_\gamma^\beta f(t)\,dt.$$

iii) The function $G : [a,b] \to \mathbb{X}$ given by

$$G(x) = \int_a^x f(t)\,dt \qquad \text{for any } x \in [a,b]$$

is continuous. In particular, $G(a) = 0$.

iv) Let $\mathbb{Y}$ be another Banach space. Let $T \in \mathcal{B}(\mathbb{X}, \mathbb{Y})$ the bounded linear operators from $\mathbb{X}$ to $\mathbb{Y}$. Then $Tf \in \mathcal{A}([a,b], \mathbb{Y})$ and moreover

$$T\left(\int_a^b f(t)\,dt \right) = \int_a^b (Tf)(t)\,dt$$

v) The function $g : [a,b] \to [0,\infty)$ given by $g(t) = \|f(t)\|$ satisfies $g \in \bar{A}([a,b], \mathbb{R})$ and moreover

$$\left\| \int_a^b f(t)dt \right\| \leq \int_a^b \|f(t)\|dt$$

<u>Note</u>: The 1st statement in v) above shows that $g(t) = \|f(t)\|$ is Riemann integrable in the traditional sense of real-valued functions. In this case, the bound in v) estimates the Banach space-valued Riemann integral in terms of a real-valued Riemann integral.

<u>On the Fundamental Theorem of Calculus</u>

We now prove the fundamental result relating differentiation and integration for Banach space valued functions.

<u>Proposition</u> Let $f \in C([a,b], X)$ for which

$f'(t) = 0$ for all $t \in (a,b)$.

Then f is constant on $[a,b]$.

The proof here is different from the proof when the function f is real valued.

proof of proposition

We begin by observing a consequence of f having zero derivative.

Let $t_0 \in (a,b)$. Note that for any $t \in (a,b) \setminus \{t_0\}$

$$\| f(t) - f(t_0) \| = \| f(t) - f(t_0) - f'(t_0)(t-t_0) \|$$

$$= |t-t_0| \cdot \left\| \frac{f(t)-f(t_0)}{t-t_0} - f'(t_0) \right\|$$

Thus for any $\varepsilon > 0$, there is a $\delta = \delta(t_0) > 0$ for which

$$(**) \quad \| f(t) - f(t_0) \| \leq \varepsilon |t - t_0| \qquad \text{for all } |t-t_0| < \delta .$$

Here we have used that $f'(t_0)$ exists for all $t_0 \in (a,b)$.

Now, let $\alpha \in (a,b)$ and take $\varepsilon > 0$.

Define

$$t_* := \sup \{ t \in [\alpha, b] : \| f(t) - f(\alpha) \| \leq \varepsilon \cdot (t - \alpha) \}.$$

We claim that for every choice of $\alpha \in (a,b)$ and $\varepsilon > 0$, $t_* = b$. Given this, for any $\alpha \in (a,b)$

$$\| f(t) - f(\alpha) \| \leq \varepsilon \,(t-\alpha) \qquad \text{for all } t \in [\alpha, b]$$

and any choice of $\varepsilon > 0$. We conclude

$$f(t) = f(\alpha) \qquad \text{for all } t \in [\alpha, b].$$

In particular, $f(\alpha) = f(b)$ for all $\alpha \in (a,b)$.
Since f is continuous, $f(a) = f(b)$ as well and f is constant!

we need only show that for each choice of $\alpha \in (a,b)$ and $\varepsilon > 0$, $t_* = b$.

Note: By $(**)$ it is clear that $t_* > \alpha$.

By way of contradiction, suppose $t_* < b$.

In this case, since t_* is defined as a supremum and f is continuous, it is clear that

$$\| f(t_*) - f(\alpha) \| \leq \varepsilon (t_* - \alpha)$$

In this case, it is also clear that $t_* \in (a,b)$ and so $(**)$ applies. Thus for the above choice of $\varepsilon > 0$, there is a $\delta = \delta(t_*) > 0$ for which

$$\| f(t) - f(t_*) \| \leq \varepsilon (t - t_*) \quad \text{whenever} \quad |t - t_*| < \delta$$

Take any $t \in (t_*, b]$ with $|t - t_*| < \delta$.

Then

$$\begin{aligned} \| f(t) - f(\alpha) \| &\leq \| f(t) - f(t_*) \| + \| f(t_*) - f(\alpha) \| \\ &\leq \varepsilon (t - t_*) + \varepsilon (t_* - \alpha) \\ &= \varepsilon (t - \alpha) \end{aligned}$$

Since $t > t_*$, this contradicts the definition of t_* as the supremum of all such t. This completes the proof.

We can now prove the main result.

Theorem (The Fundamental Theorem of Calculus)

Let $\mathbb{X}$ be a Banach space and $[a,b] \subset \mathbb{R}$ a compact interval.

1) If $f \in C([a,b], \mathbb{X})$, then $G : [a,b] \to \mathbb{X}$ defined by

$$G(x) = \int_a^x f(t)\,dt \qquad \text{for all } x \in [a,b]$$

satisfies

$$G'(x) = f(x) \qquad \text{for all } x \in (a,b).$$

Note: with "one-sided" derivatives, this can be extended to the end points $x = a$ and $x = b$.

2) If $f \in C([a,b], \mathbb{X})$ and $f' \in C([a,b], \mathbb{X})$, then

$$\int_a^b f'(t)\,dt = f(b) - f(a).$$

Proof:

1) Let $x \in (a,b)$ and $h \neq 0$ satisfy $x+h \in [a,b]$.

Then

$$\| G(x+h) - G(x) - h \cdot f(x) \| = \left\| \int_x^{x+h} (f(t) - f(x))\,dt \right\|$$

$$\leq \int_x^{x+h} \| f(t) - f(x) \|\,dt$$

$$\leq |h| \cdot \sup_{t \in [x, x+h]} \| f(t) - f(x) \|$$

Use Lemma ii) and def. of integral

Lemma v

Since f is continuous, for each $\varepsilon > 0$ there is a $\delta > 0$ for which

$$\| f(t) - f(x) \| \leq \varepsilon \quad \text{whenever} \quad |t - x| < \delta .$$

Let $h \neq 0$ be any number with $x + h \in [a,b]$ and $|h| < \delta$

In this case, we conclude the claim in 1) above.

2) Consider the function $H : [a,b] \to \underline{X}$ defined by

$$H(x) = \int_a^x f'(t)\,dt - f(x)$$

By assumption and Lemma iii), H is continuous.

In fact, by assumption and part 1) above, H is differentiable with

$$H'(x) = f'(x) - f'(x) = 0 \qquad \text{for all } x \in (a,b).$$

By our proposition, H is constant.

In this case,

$$H(b) = H(a) \iff \int_a^b f'(t)\,dt - f(b) = \int_a^a f'(t)\,dt - f(a)$$

$$\iff \int_a^b f'(t)\,dt = f(b) - f(a)$$

For the last claim, we used Lemma iii).

As a final result, we mention the mean value inequality.

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<u>Corollary</u> Suppose $f \in C([a,b], X)$ and $f' \in C([a,b], X)$.

<u>Then</u>

$$\| f(b) - f(a) \| \le \int_a^b \| f'(t) \| \, dt \le (b-a) \cdot \| f' \|_\infty .$$

<u>Proof:</u>

By the Theorem above

$$f(b) - f(a) = \int_a^b f'(t) \, dt$$

$$\Rightarrow \quad \| f(b) - f(a) \| = \Big\| \int_a^b f'(t) \, dt \Big\|$$

$$\overset{\text{Lemma } v)}{\le} \int_a^b \| f'(t) \| \, dt$$

$$\le (b-a) \cdot \| f' \|_\infty$$