Lecture 18

Last class we discussed locality, or the lack thereof, for the finite-volume Heisenberg dynamics associated to an interaction. Today we discuss quasi-locality.

Recall:

A regularity condition on (Γ, d).

We will say that a non-increasing function $F : [0, \infty) \to (0, \infty)$ is an F-function on (Γ, d) if

i) F is uniformly summable on Γ, i.e.

$$\|F\| = \sup_{x \in \Gamma} \sum_{y \in \Gamma} F(d(x,y)) < \infty$$

and

ii) F satisfies a convolution condition, i.e.

$$C_F = \sup_{x,y \in \Gamma} \sum_{z \in \Gamma} \frac{F(d(x,z))\, F(d(z,y))}{F(d(x,y))} < \infty.$$

We will say that a metric space (Γ, d) is regular if there is an F-function on (Γ, d).

Some comments

1) Since an F-function has strictly positive values, the summability condition i) immediately implies that Γ contains at most countably many points.

2) From the convolution condition ii), we know that for any $x, y \in \Gamma$

$$\sum_{z \in \Gamma} F(d(x,z)) \cdot F(d(z,y)) \leq C_F \, F(d(x,y)).$$

This bound will be useful in our estimates.

3) We checked last class that $(\Gamma, d) = (\mathbb{Z}^\nu, |\cdot|)$ for any $\nu \geq 1$ is regular. In fact, we showed that

$$F(r) = \frac{1}{(1+r)^{\nu+1}} \quad \text{for } r \geq 0$$

is an F-function on $(\mathbb{Z}^\nu, |\cdot|)$. Here

$$\|F\| = \sum_{z \in \mathbb{Z}^\nu} \frac{1}{(1+|z|)^{\nu+1}}$$

and

$$C_F \leq 2^{\nu+2} \|F\|.$$

4) We also showed that if F_0 is an F-function on (Γ, d), then for any $a \geq 0$

$$F_a(r) = e^{-ar} \cdot F_0(r) \qquad \text{for } r \geq 0$$

defines an F-function on (Γ, d). Moreover,

$$\|F_a\| \leq \|F_0\| \quad \text{and} \quad C_{F_a} \leq C_{F_0}.$$

We now use these F-functions to describe the "decay of interactions".

Let (Γ, d) be a regular metric space and let F be an F-function on (Γ, d). ~~Take~~ Let Φ be an interaction on (Γ, d) and denote by

$$\|\Phi\|_F = \sup_{x,y \in \Gamma} \sum_{\substack{X \subset \Gamma: \\ x,y \in X}} \frac{\|\Phi(X)\|}{F(d(x,y))}$$

Take

$$B_F(\Gamma) = \{\Phi \mid \Phi \text{ is an interaction on } (\Gamma, d) \text{ and } \|\Phi\|_F < \infty\}.$$

One can check that $B_F(\Phi)$ is a Banach space of interactions with norm $\|\cdot\|_F$. <u>Note</u>: Arithmetic for interactions is defined as with all functions:

$$(\Phi + \Psi)(X) = \Phi(X) + \Psi(X) \qquad \text{for all } X \subset \Gamma \text{ finite.}$$
$$(\lambda \Phi)(X) = \lambda \Phi(X)$$

For any $\Phi \in \mathcal{B}_F(\Gamma)$, one has that:

For any $x, y \in \Gamma$

$$\sum_{\substack{X \subset \Gamma: \\ x, y \in X}} \|\Phi(X)\| \leq \|\Phi\|_F \cdot F(d(x,y))$$

In this sense, we say that the F-function F governs the decay of the interaction Φ. In words, for any pair of sites $x, y \in \Gamma$, the sum (of the norm) of all interaction terms that contain x and y is bounded above by a constant (namely $\|\Phi\|_F$) multiplied by $F(d(x,y))$.

We can now state an important quasi-locality bound for the Heisenberg dynamics associated to any $\Phi \in \mathcal{B}_F(\Gamma)$.

<u>Theorem (Lieb-Robinson Bound)</u>

Let (Γ, d) be a regular metric space, F be an F-function on (Γ, d), and $\Phi \in \mathcal{B}_F(\Gamma)$. For any finite sets $X, Y \subset \Gamma$ with $X \cap Y = \emptyset$ and any finite set $\Lambda \subset \Gamma$ with $X \cup Y \subset \Lambda$, one has that:

$$\|[\tau_t^{\Lambda,\Phi}(A), B]\| \leq \frac{2\|A\| \cdot \|B\|}{C_F}\left(e^{2\|\Phi\|_F C_F |t|} - 1\right) \sum_{x \in X} \sum_{y \in Y} F(d(x,y))$$

for all $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$, and $t \in \mathbb{R}$.

Some Comments on this bound

(1) The estimate on the right-hand-side above is independent of the finite volume $\Lambda \subset \Gamma$ on which the Heisenberg dynamic is defined. This is an indication that such a bound will persist in the thermodynamic limit (i.e. the limit of $\Lambda \to \Gamma$). We will discuss this after the proof.

(2) Since this Heisenberg dynamics $\tau_t^{\Lambda,\Phi}$ is an automorphism of $\mathcal{A}_\Lambda$, it is clear that

$$\|[\tau_t^{\Lambda,\Phi}(A), B]\| \leq 2 \cdot \|\tau_t^{\Lambda,\Phi}(A)\| \cdot \|B\| = 2 \cdot \|A\| \cdot \|B\|$$

In this case, the commutator above is bounded by a constant (i.e. $2\|A\| \cdot \|B\|$) uniformly in time t. Since the time dependence in the estimate of the Lieb-Robinson bound grows exponentially in $|t|$, it is clear that Lieb-Robinson bounds will only be "useful" for small times. In the next comment, we quantify this notion of "small times" in a special case.

3. As we indicated before, if (Γ, d) is regular and F_0 is an F-function on (Γ, d), then $F_a(r) = e^{-ar} F_0(r)$ is also an F-function on (Γ, d). Let us now consider the case that $\Phi \in B_{F_a}(\Gamma)$ for some $a > 0$. In this case, the Lieb-Robinson bound becomes:

$$\| [\tau_t^{\Lambda,\Phi}(A), B] \| \leq \frac{2\|A\| \cdot \|B\|}{C_{F_a}} \left(e^{2\|\Phi\|_{F_a} C_{F_a} |t|} - 1 \right) \sum_{\substack{x \in X \\ y \in Y}} F_a(d(x,y))$$

$$\leq \frac{2 \cdot \|A\| \cdot \|B\|}{C_{F_a}} \cdot e^{2\|\Phi\|_{F_a} C_{F_a} |t|} \cdot e^{-a d(X,Y)} \cdot \sum_{\substack{x \in X \\ y \in Y}} F_0(d(x,y))$$

$$\leq \frac{2 \cdot \|A\| \cdot \|B\|}{C_{F_a}} \cdot \|F_0\| \cdot |X| \cdot e^{-a(d(X,Y) - v_a |t|)}$$

where $v_a = \dfrac{2\|\Phi\|_{F_a} C_{F_a}}{a} > 0$.

Here we:

1. Dropped the -1 in the 1st estimate.
2. We wrote: $F_a(d(x,y)) = e^{-a d(x,y)} F_0(d(x,y))$
$$\leq e^{-a d(X,Y)} \cdot F_0(d(x,y)) \quad \text{for all } x \in X \text{ and } y \in Y$$

Since $d(X,Y) = \min\{ d(x,y) \mid x \in X \text{ and } y \in Y \}$.

(3) We estimated

$$\sum_{\substack{x\in X\\ y\in Y}} F_a(d(x,y)) = \sum_{x\in X} \sum_{y\in Y} F_a(d(x,y)) \le \sum_{x\in X} \|F_a\| = \|F_a\|\cdot|X| \leftarrow \text{the cardinality of } X\subset\Gamma.$$

The resulting bound is this:

$$\|[\tau_t^{\Lambda,\Phi}(A), B]\| \le \frac{2\cdot\|A\|\cdot\|B\|}{C_{F_a}} \|F_a\|\cdot|X|\, e^{-a(d(X,Y)-v_a|t|)}$$

with some number $v_a>0$. This number v_a is often called the Lieb-Robinson velocity associated to Φ.

<u>Note that</u>: If $|t| < \dfrac{d(X,Y)}{v_a}$, then the bound above is exponentially small.

We took A and B to be observables with disjoint supports X and Y. In this case, we know that

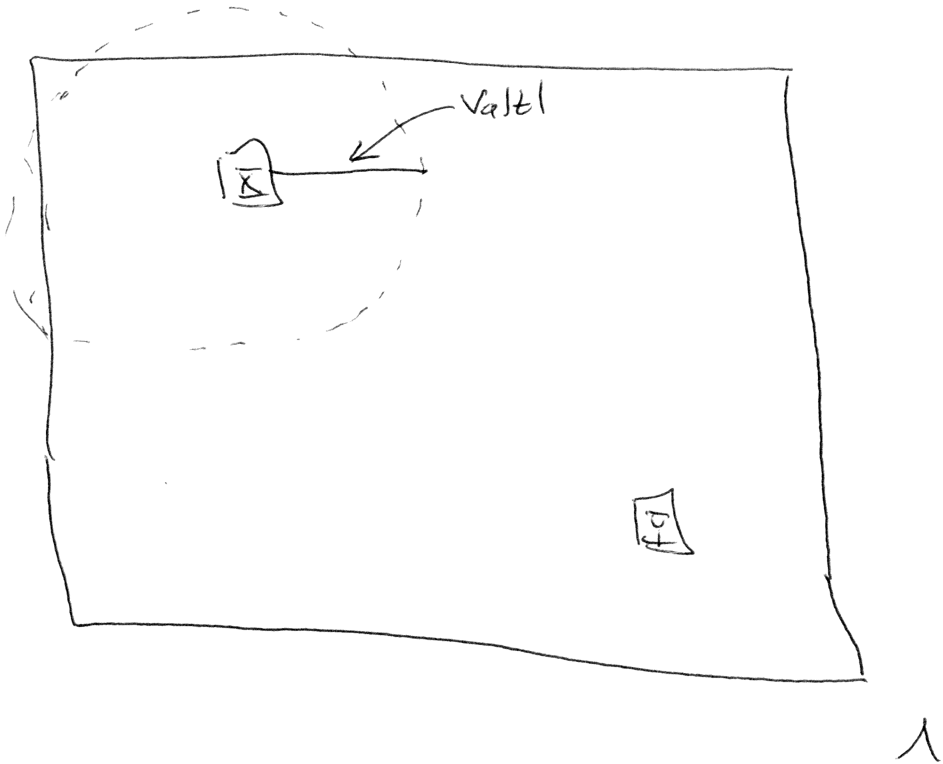
$$0 = [A,B] = [\tau_0^{\Lambda,\Phi}(A), B].$$

The Lieb-Robinson bound shows that for small times $|t|$, quantified by $|t| < \dfrac{d(X,Y)}{v_a}$, the norm of this commutator is still small in the sense that $e^{-a(d(X,Y)-v_a|t|)}$ is small.

This estimate is interpreted by saying that the corresponding Heisenberg dynamics is "quasi-local".

One can visualize this estimate in a number of ways.

Consider:



For any real number $n \geq 0$, denote by

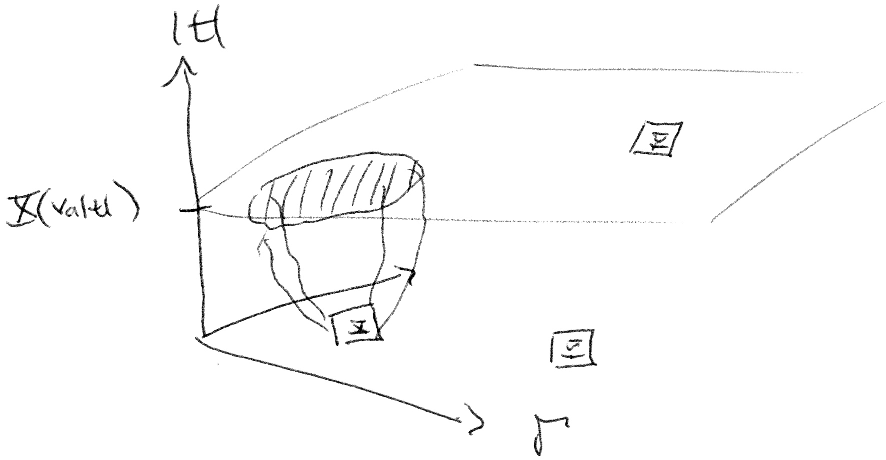
$$X(n) = \{z \in \Gamma : \text{there is } x \in X \text{ with } d(x,z) \leq n\}$$

Fix $X \subset \Gamma$ finite and $t \in \mathbb{R}$. The Lieb Robinson bound shows that the commutator of $\tau_t^{\Lambda,\Phi}(A)$ with B will be exponentially small if $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$ where $Y \subset \Lambda / X(v_a|t|)$. In this sense, the support of $\tau_t^{\Lambda,\Phi}(A)$ is essentially contained in $X(v_a|t|)$. Thus:

- $\tau_t^{\Lambda,\Phi}(A)$ is essentially local
- v_a governs the propagation rate associated to the support of $\tau_t^{\Lambda,\Phi}(A)$

The following "light-cone" picture is also useful.

Consider $X \subset \Gamma$ finite, $t \in \mathbb{R}$, and $A \in \mathfrak{A}_X$.



At time $t>0$, observables with support far away from $X(v_a|t|)$ have small commutators with $\tau_t^{\Lambda,\Phi}(A)$.

Thus $X(v_a|t|)$ defines a "light-cone" for the essential support of $\tau_t^{\Lambda,\Phi}(A)$.

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4) <u>Warning</u>:

Although the phrase "Lieb-Robinson velocity" is often used in the literature, this can be somewhat misleading. The bound proven by Lieb and Robinson is an upper estimate. It gives an upper bound on the maximum rate of propagation in the system. This bound does not imply the existence of observables whose support propagates exactly at this rate. Such a statement would require a lower bound and these are not generally known. (There are known results for 1-dimensional toy models.)