Lecture 19

On Time-Dependent Interactions

Let $I \subset \mathbb{R}$ be an open interval.

Let $\mathcal{H}$ be a Hilbert space and $H : I \to B(\mathcal{H})$ satisfy

i) $H(t)^* = H(t)$ for all $t \in I$, i.e. H is pointwise self-adjoint on I

ii) $t \mapsto H(t)$ in norm-continuous.

Our goal is to describe solutions of the initial value problem:

$$(*) \qquad \frac{d}{dt}\psi(t) = -iH(t)\psi(t) \quad \text{for } t \in I \quad \text{with } \psi(t_0) = \psi_0 \in \mathcal{H}$$

for some $t_0 \in I$.

$(*)$ is often called the time-dependent Schrödinger equation.

To find this solution, let us assume it exists and then investigate its properties.

Suppose $\psi : I \to \mathcal{H}$ is a solution of $(*)$. Then for any $t \in I$

$$\psi(t) - \psi(t_0) = \int_{t_0}^{t} \frac{d}{ds}\psi(s)\,ds = \int_{t_0}^{t} -iH(s)\psi(s)\,ds$$

Let us set $t_0 = 0$ for convenience.

Since $\psi(t_0) = \psi(0) = \psi_0$, we conclude that

$$\psi(t) = \psi_0 + (-i)\int_0^t H(s_1)\psi(s_1)\,ds_1, \qquad \text{Now iterate:}$$

$$= \psi_0 + (-i)\int_0^t H(s_1)\left[\psi_0 + (-i)\int_0^{s_1} H(s_2)\psi(s_2)\,ds_2\right] ds_1$$

$$= \psi_0 + (-i)\int_0^t H(s_1)\psi_0\,ds_1 + (-i)^2\int_0^t H(s_1)\int_0^{s_1} H(s_2)\left[\psi_0 + (-i)\int_0^{s_2} H(s_3)\psi(s_3)\,ds_3\right] ds_2\,ds_1$$

$$= \psi_0 + (-i)\int_0^t H(s_1)\psi_0\,ds_1 + (-i)^2\int_0^t H(s_1)\int_0^{s_1} H(s_2)\psi_0\,ds_2\,ds_1 + (-i)^3 \cdots$$

This expansion motivates the following definition.

For each $t \in I$ set

$$U(t,0) = \mathbb{1} + \sum_{n=1}^{\infty} (-i)^n \int_0^t \cdots \int_0^{s_{n-1}} H(s_1)H(s_2)\cdots H(s_n)\,ds_n\,ds_{n-1}\cdots ds_1 .$$

The right-hand-side above is called the <u>Dyson Series</u> associated to the time dependent Hamiltonian H. In the physics literature, this is often called a "time-ordered exponential" although it is not generally an exponential...

Our goal is to now check several important facts.

1) For each $t \in I$, $U(t,0) \in B(\mathcal{H})$ is well-defined.

2) $\frac{d}{dt}U(t,0) = -iH(t)U(t,0)$ for all $t \in I$

3) $U(t,0)$ is unitary for all $t \in I$.

To check that these properties are true, we begin with some additional notation. For each $n \geq 1$, define a function $U_n : I \to B(H)$ inductively by setting

$$U_1(t) = H(t)$$
$$U_2(t) = H(t) \cdot \int_0^t U_1(s)\,ds$$
$$\vdots$$
$$U_n(t) = H(t) \int_0^t U_{n-1}(s)\,ds \qquad \text{for all } n \geq 2.$$

By assumption, U_1 is continuous. Since U_1 is continuous, it is integrable. Moreover, its integral is also continuous. In this case, U_2 is continuous as the product of two continuous functions. By induction, U_n is continuous for all $n \geq 1$.

Note that the Dyson series of interest can be re expressed as:

$$U(t,0) = \mathbb{1} + \sum_{n=1}^{\infty} (-i)^n \int_0^t U_n(s)\,ds.$$

To show that $U(t,0)$ is a well-defined element of $B(H)$, we show that the sequence of partial sums is Cauchy.

~~Let~~ For each $N \geq 1$, let

$$S_N(t) = \mathbb{1} + \sum_{n=1}^{N} (-i)^n \int_0^t U_n(s)\,ds.$$

As we have checked, $S_N(t) \in B(H)$ for all $t \in I$. S_N is, in fact, a continuous function.

For any $1 \le M < N < \infty$, one has that

$$S_N(t) - S_M(t) = \sum_{n=M+1}^{N} (-i)^n \int_0^t u_n(s)\, ds \quad \text{for any } t \in I.$$

In this case,

$$\| S_N(t) - S_M(t) \| \le \sum_{n=M+1}^{N} \int_0^t \| u_n(s) \|\, ds$$

$$\le \sum_{n=M+1}^{N} \int_0^t \cdots \int_0^{s_{n-1}} \|H(s_1)\| \cdot \|H(s_2)\| \cdot \cdots \|H(s_n)\|\, ds_n \cdots ds_1$$

$$= \sum_{n=M+1}^{N} \frac{\left(\int_0^t \|H(s)\| ds \right)^n}{n!}$$

$$\longrightarrow 0 \quad \text{as } N, M \to \infty.$$

Here we have used the following homework problem.

Let $f : [0,\infty) \to [0,\infty)$ be a continuous function.

Show that for any $t \in [0,\infty)$ and each $n \ge 1$

$$\int_0^t \cdots \int_0^{s_{n-1}} f(s_1) \cdots f(s_n)\, ds_n \cdots ds_1 = \frac{\left(\int_0^t f(s) ds \right)^n}{n!}.$$

The above estimate also shows that this convergence is uniform for t in compact subsets (containing $0 = t_0 \ldots$).

Since $B(H)$ is a Banach space, for each $t \in I$, the limit:

$$\lim_{N\to\infty} S_N(t) = \lim_{N\to\infty}\left[\mathbb{1} + \sum_{n=1}^{N} (-i)^n \int_0^t U_n(s)\,ds \right]$$

$$= \mathbb{1} + \sum_{n=1}^{\infty} (-i)^n \int_0^t U_n(s)\,ds$$

$$= U(t,0) \qquad \text{exists!}$$

Our estimates also imply that:

$$\|U(t,0)\| \le 1 + \sum_{n=1}^{\infty} \frac{\left(\int_0^t \|H(s)\|\,ds\right)^n}{n!} = e^{\int_0^t \|H(s)\|\,ds} \quad \text{for all } t \in I.$$

Note further that for any $N \ge 1$,

$$\frac{d}{dt} S_N(t) = 0 + \sum_{n=1}^{N} (-i)^n \frac{d}{dt} \int_0^t U_n(s)\,ds$$

$$= \sum_{n=1}^{N} (-i)^n U_n(t)$$

$$= (-i) H(t) + \sum_{n=2}^{N} (-i)^n H(t) \int_0^t U_{n-1}(s)\,ds$$

$$= (-i) H(t) S_{N-1}(t)$$

Taking limits as $N \to \infty$, we conclude that the series whose terms ~~contains~~ are the derivatives (of the terms defining $U(t,0)$)

converges to the expected derivative of $U(t,0)$; namely (6)

$$\sum_{n=1}^{\infty} (-i)^n \frac{d}{dt} \int_0^t u_n(s)\,ds = -i\,H(t)\,U(t,0).$$

We need only prove that the left-hand-side is actually the derivative of $U(t,0)$. The "old calculus" result justifying term-by-term differentiation of a series applies here and completes the proof of this fact. I'll just state it briefly (for the sake of completeness) and leave the proof to the interested reader...

<u>Theorem</u> (from calculus)

Suppose that for each $k \geq 1$, $u_k : [a,b] \to \mathbb{R}$ is a function with a continuous derivative on $[a,b]$. (Here we use one-sided notions at the end points.) Suppose further that

i) The series $\sum_{k=1}^{\infty} u_k(x_0)$ converges at some $x_0 \in [a,b]$.

ii) The series $\sum_{k=1}^{\infty} u_k'(x)$ converges uniformly on $[a,b]$ to $f(x)$ on $[a,b]$, i.e. $f(x) = \sum_{k=1}^{\infty} u_k'(x)$ for all $x \in [a,b]$.

Then the series $\sum_{k=1}^{\infty} u_k(x)$ converges for all $x \in [a,b]$ and the function $F(x) = \sum_{k=1}^{\infty} u_k(x)$ for all $x \in [a,b]$ is differentiable with

$$F'(x) = f(x) \text{ for all } x \in [a,b].$$

Let us now check that $U(t,0)$ is unitary for all $t \in I$.

We just checked that $U(t,0)$ satisfies the $B(\mathcal{H})$ valued initial value problem

$$\frac{d}{dt} U(t,0) = -iH(t)\,U(t,0) \quad \text{for all } t \in I \quad \text{with } U(0,0) = \mathbb{1}.$$

By taking adjoints, we conclude that

$$\frac{d}{dt} U(t,0)^* = i\,U(t,0)^* H(t) \quad \text{for all } t \in I \quad \text{with } U(0,0)^* = \mathbb{1}$$

In this case,

$$\frac{d}{dt} U(t,0)^* U(t,0) = i\,U(t,0)^* H(t) U(t,0) - i\,U(t,0)^* H(t) U(t,0)$$
$$= 0 \quad \text{for all } t \in I.$$

Thus $U(t,0)^* U(t,0)$ is constant for all $t \in I$.

$\mathbb{1} = U(0,0)^* U(0,0) = U(t,0)^* U(t,0) \Rightarrow U(t,0)$ is unitary for all $t \in I$.

Note that: For any $\psi_0 \in \mathcal{H}$,

$\psi(t) = U(t,0)\psi_0$

is easily checked to be a solution of $(*)$. By unitarity,

$\|\psi(t)\| = \|\psi_0\|$ for all $t \in I$.

We will use a Gronwall argument (in homework) to prove that this solution is unique.

The Heisenberg dynamics

For any $s, t \in I$, the quantity

$$U(t,s) = U(t,0)\, U(s,0)^*$$

is easily checked to be a two-parameter family of unitaries.

One also checks that

$$U(t,s)\, U(s,r) = U(t,r) \qquad \text{for all } r,s,t \in I$$

and hence this family of unitaries satisfies the group property.

The Heisenberg dynamics associated to the time dependent Hamiltonian H is :

$$\tau_{t,s}(A) = U(t,s)^* A\, U(t,s) \qquad \text{for all } A \in B(\mathcal{H}) \text{ and } s,t \in I.$$

This is the time-evolution from time s to time t.

One checks that

$$\frac{d}{dt} \tau_{t,s}(A) = i\, \tau_{t,s}\left([H(t), A] \right)$$

similar to the previous case. Also, since $U(t,s)$ is unitary, and $A \in B(\mathcal{H})$

$$\|\tau_{t,s}(A)\| = \|A\| \qquad \text{for all } s,t \in I$$

and this $\tau_{t,s}$ is an automorphism of $B(\mathcal{H})$.

A Norm-Preservation Lemma

The following lemma will be useful in our proof of the Lieb-Robinson bound.

<u>Lemma</u> Let $I \subset \mathbb{R}$ be an open interval and $\mathcal{H}$ a complex Hilbert space. Let $A, B: I \to \mathcal{B}(\mathcal{H})$ satisfy

i) $A(t)^* = A(t)$ for all $t \in I$.

ii) $t \mapsto A(t)$ and $t \mapsto B(t)$ are norm-continuous.

Then there is a unique solution $y: I \to \mathcal{B}(\mathcal{H})$ of the initial value problem

$$\frac{d}{dt} y(t) = -i [A(t), y(t)] + B(t) \qquad \text{for } t \in I$$

with $y(t_0) = y_0 \in \mathcal{B}(\mathcal{H})$ for some $t_0 \in I$.

Moreover, the norm of this solution satisfies:

$$\|y(t)\| \leq \|y_0\| + \int_{\min(t_0,t)}^{\max(t_0,t)} \|B(s)\| \, ds \qquad \text{for all } t \in I.$$

Proof:

Our proof is constructive.

Let $U(t,t_0)$ be the Dyson series associated to A, i.e.

$U(t,t_0)$ is the unique unitary which solves

$$\frac{d}{dt}U(t,t_0) = -i\,A(t)U(t,t_0) \quad \text{for } t\in I \text{ with } U(t_0,t_0)=\mathbb{1}.$$

Consider the function $y: I \to B(\mathcal{H})$ defined by setting

$$y(t) = U(t,t_0)\left(y_0 + \int_{t_0}^{t} U(s,t_0)^* B(s)\, U(s,t_0)\,ds\right)U(t,t_0)^*$$

for all $t\in I$.

It is clear that:

$$y(t_0) = U(t_0,t_0)\left(y_0 + \int_{t_0}^{t_0} U(s,t_0)^* B(s) U(s,t_0)\,ds\right)U(t_0,t_0)^* = y_0$$

and moreover,

$$y'(t) = -i\,A(t)y(t) + U(t,t_0)\,U(t,t_0)^* B(t)\,U(t,t_0)\,U(t,t_0)^*$$

$$+\,i\,y(t)A(t)$$

$$= -i\,[A(t), y(t)] + B(t) \qquad \text{for all } t\in I.$$

Thus y is a well-defined solution of the initial value problem.

Moreover,

$$\|y(t)\| \leq \|y_0\| + \int_{\min(t_0,t)}^{\max(t_0,t)} \|B(s)\|\,ds \qquad \text{as claimed.}$$

Uniqueness, via ~~gron~~ Gronwall, will be proven in homework.