# II Quantum Spin Systems

## 2.2 Observables:

By $\mathcal{H}$, we will denote the complex Hilbert space of "states" associated to a quantum system.

By $\mathcal{B}(\mathcal{H})$, we will denote the collection of all bounded linear operators on $\mathcal{H}$.
This means that $A \in \mathcal{B}(\mathcal{H})$ if

i) $A$ is linear, i.e.

$$A(\alpha x + \beta y) = \alpha A(x) + \beta A(y) \qquad \text{for all } x, y \in \mathcal{H} \text{ and } \alpha, \beta \in \mathbb{C}$$

and

ii) $A$ is bounded, i.e. there is some $0 \le C_A < \infty$ for which

$$\| Ax \| \le C_A \cdot \|x\| \qquad \text{for all } x \in \mathcal{H}.$$

Two comments

- The quantity $\|\cdot\|$ is the norm induced by the inner product, i.e. $\|x\| = \sqrt{\langle x, x \rangle}$ for all $x \in \mathcal{H}$.

- The notion of boundedness of operators is different that the notion of boundedness of functions:

<u>Recall</u>: $f : \mathbb{R} \to \mathbb{C}$ is bounded if

$$\sup_{x \in \mathbb{R}} |f(x)| < \infty .$$

---

We will refer to $B(\mathcal{H})$ as the <u>algebra of observables</u> associated to $\mathcal{H}$. More generally, we will use the notation $\mathcal{Q}$ for an algebra of observables associated to a quantum system. (More on this later...)

---

An important example for our class is as follows:

<u>Ex</u> Let $d \geq 2$ be an integer.

Take $\mathcal{H} = \mathbb{C}^d$.

$\mathcal{H}$ is a Hilbert space when equipped with the standard inner product

$$\langle x, y \rangle = \sum_{i=1}^{d} \overline{x_i}\, y_i \quad \text{for all } x, y \in \mathcal{H}$$

<u>Here</u> $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{C}^d$

i.e. $x_j \in \mathbb{C}$ $\forall\, 1 \leq j \leq$

Ex (cont.)

In this case, $B(H) \cong M_d$.

Here $M_d$ is the collection of all $d \times d$ matrices with complex entries.

The statement $B(H) \cong M_d$ means that there is a 1 to 1 relationship between these two sets. This requires that we fix a basis, but...

In the future, I will simply write $B(H) = M_d$ and there will be no confusion.

---

Let us establish some notation and terminology. Let $d \geq 2$, consider $H = \mathbb{C}^d$ and $B(H) = M_d$.

Each $A \in M_d$ is a matrix i.e.

$$A = \{a_{ij}\}_{i,j=1}^{d} \quad \text{with} \quad a_{ij} \in \mathbb{C} \quad \text{for all } 1 \leq i,j \leq d.$$

To each $A \in M_d$, there is a unique matrix $A^* \in M_d$ (called the hermitian conjugate of $A$) defined by

$$A^* = \{a^*_{ij}\}_{i,j=1}^{d} \quad \text{with} \quad a^*_{ij} = \overline{a_{ji}} \quad \leftarrow \text{the complex conjugate.}$$

One readily checks that, with respect to the standard inner product on $\mathbb{C}^d$, the relation

$$(*) \qquad \langle Ax, y\rangle = \langle x, A^* y\rangle$$

holds for all $x, y \in \mathbb{C}^d$ and $A \in M_d$.

---

For more general Hilbert spaces $\mathcal{H}$, to each $A \in B(\mathcal{H})$ one may associate a unique $A^* \in B(\mathcal{H})$ by requiring that $A^*$ is a linear map satisfying:

$$\langle Ax, y\rangle = \langle x, A^* y\rangle \quad \text{for all } x, y \in \mathcal{H}.$$

This observable $A^* \in B(\mathcal{H})$ is called the adjoint of $A$.

---

Moreover, a matrix $A \in M_d$ is said to be hermitian if $A^* = A$.

$\uparrow$ hermitian conjugate.

More generally, an observable $A \in B(\mathcal{H})$ is said to be self-adjoint if $A^* = A$.

$\uparrow$ the adjoint

For the purposes of this class, I will only use the terminology adjoint and self-adjoint (even in the context of matrices.)

---

<u>Remark</u>: In much of the physics literature, the term "observable" is usually reserved for self-adjoint $A \in B(\mathcal{H})$. As we will see, $B(\mathcal{H})$ has the mathematical structure of an algebra, and this structure will be convenient for certain arguments. For this reason, we will refer to all elements of $B(\mathcal{H})$ as observables.

---

Back to our main example.

Let $d \geq 2$. $\mathcal{H} = \mathbb{C}^d$ and $B(\mathcal{H}) = M_d$.

As we have indicated previously, there is a standard inner product on $\mathbb{C}^d$:

$$\langle x, y \rangle = \sum_{i=1}^{d} \overline{x_i} y_i \qquad \text{for all } x, y \in \mathbb{C}^d$$

The corresponding norm on $\mathbb{C}^d$ is

$$\|x\| = \sqrt{\langle x,x\rangle} = \sqrt{\sum_{i=1}^{d} |x_i|^2} \,.$$

It will also be useful to observe that $B(H) = M_d$ has a standard inner product as well. The map

$$\langle A, B\rangle_{HS} = \mathrm{Tr}[A^*B] \quad \text{for all } A, B \in M_d$$

defines the Hilbert-Schmidt inner product on $M_d$.

Here the map $\mathrm{Tr} : M_d \to \mathbb{C}$ is the trace i.e.

$$\mathrm{Tr}[A] = \sum_{i=1}^{d} a_{ii} \quad \text{where } A = \{a_{ij}\}_{i,j=1}^{d} \in M_d.$$

Properties of $\langle \cdot, \cdot \rangle_{HS}$ will be investigated in the 1st Homework assignment.

The corresponding norm is:

$$\|A\|_{HS} = \sqrt{\langle A, A\rangle_{HS}} = \sqrt{\mathrm{Tr}[A^*A]} \,.$$

This shows that both $H = \mathbb{C}^d$ and $B(H) = M_d$ are complex, finite dimensional inner product spaces and hence Hilbert spaces as well.

An important example concerns the case of $d=2$.

Let $\mathcal{H} = \mathbb{C}^2$ and $B(\mathcal{H}) = M_2$.

A standard basis for $\mathcal{H} = \mathbb{C}^2$ is $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$.

A standard basis for $B(\mathcal{H}) = M_2$ is

$$\sigma^0 = \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\left.\begin{aligned} \sigma^1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma^2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ \sigma^3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}\right\}$$

These are the Pauli-matrices

(often called the Pauli-spin matrices)

With respect to $\frac{1}{2}\langle \cdot, \cdot \rangle_{HS}$, the collection $\{\sigma^0, \sigma^1, \sigma^2, \sigma^3\}$ form an orthonormal basis of $M_2$.

This is particularly convenient for calculations (8)

Clearly each $A \in M_2$ can be written as

$$A = \sum_{j=0}^{3} c_j \sigma^j \quad \text{with} \quad c_j = \frac{\langle \sigma^j, A \rangle_{HS}}{2}$$

using that this is an orthonormal basis. Moreover

$$\|A\|_{HS}^2 = \sum_{j=0}^{3} |c_j|^2 = \frac{1}{4} \sum_{j=0}^{3} |\langle \sigma^j, A \rangle_{HS}|^2 .$$

---

A final comment on observables.

If $\mathcal{H}$ is a Hilbert space and $\mathcal{A} = \mathcal{B}(\mathcal{H})$ is the corresponding algebra of observables, the "standard" norm of an observable is the operator norm:

$$\|A\| = \sup_{\substack{\psi \in \mathcal{H}: \\ \psi \neq 0}} \frac{\|A\psi\|}{\|\psi\|}$$

for all $A \in \mathcal{A} = \mathcal{B}(\mathcal{H})$.

## 2.3 States

Consider a quantum system with algebra of observables $\mathcal{A}$. (In general $\mathcal{A}$ will be a $C^*$-algebra; more on this later. Here it is convenient to think of $\mathcal{A} = \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.)

A state on $\mathcal{A}$ is a normalized, positive linear functional on $\mathcal{A}$. More precisely, $\omega$ is a state on $\mathcal{A}$ if

i) $\omega : \mathcal{A} \to \mathbb{C}$ is linear, ie

$$\omega(\lambda A + \mu B) = \lambda \omega(A) + \mu \omega(B) \quad \text{for all } A, B \in \mathcal{A} \text{ and } \lambda, \mu \in \mathbb{C}.$$

ii) $\omega$ is positive if $\omega(A^* A) \geq 0$ for all $A \in \mathcal{A}$.

and

iii) $\omega$ is normalized if $\omega(\mathbb{1}) = 1$.

For any $A \in \mathcal{A}$, $\omega(A)$ is referred to as the expected value of $A$ in the state $\omega$; or as the expectation of the observable $A$.

From the definition, one easily sees that

$$(**) \qquad \omega(A^*) = \overline{\omega(A)} \qquad \text{for any } A \in \mathcal{A}.$$

<u>Note</u>: This immediately implies that the expectation value of a self-adjoint observable is <u>always</u> <u>real</u>.

---

<u>Remark</u>: $(**)$ follows from:

$$\omega\big((\mathbb{1}+\lambda A)^*(\mathbb{1}+\lambda A)\big) \geq 0$$

True for any $A \in \mathcal{A}$ and $\lambda \in \mathbb{C}$.

Take $\lambda = 1$ and $\lambda = i$ and calculate imaginary parts ...

---

The <u>variance of</u> an observable $A \in \mathcal{A}$ is given by

$$\mathrm{Var}(A) = \omega\big((A - \omega(A)\mathbb{1})^*(A - \omega(A)\mathbb{1})\big)$$

$$= \omega(A^*A) - |\omega(A)|^2$$

Here you use $(**)$.

As the language indicates, there are deep relationships between quantum systems and probability theory.

There are many states...

Let $\mathcal{A} = B(\mathcal{H})$ where $\mathcal{H}$ is some Hilbert space.

Let $\psi \in \mathcal{H}$ with $\|\psi\| = 1$. (we call such vectors $\psi \in \mathcal{H}$ unit vectors)

The map $\omega_\psi : \mathcal{A} \to \mathbb{C}$ given by

$$\omega_\psi(A) = \langle \psi, A\psi \rangle \quad \text{for all } A \in \mathcal{A} = B(\mathcal{H})$$

is a state.

Note: • $\omega_\psi$ is linear ✓

• $\omega_\psi$ is positive, $\omega_\psi(A^*A) = \langle \psi, A^*A\psi \rangle = \langle A\psi, A\psi \rangle = \|A\psi\|^2 \geq 0$ ✓

• $\omega_\psi$ is normalized, $\omega_\psi(\mathbb{1}) = \langle \psi, \mathbb{1}\psi \rangle = \langle \psi, \psi \rangle = \|\psi\|^2 = 1$ (as $\psi$ is a unit vector)

States of this form are called vector states.

Note that the state $\omega_\psi$ can be written differently.

$$\omega_\psi(A) = \mathrm{Tr}[P_\psi A] \quad \text{for all } A \in \mathcal{A} = B(\mathcal{H})$$

where $P_\psi \in B(\mathcal{H})$ is the orthogonal projection onto the 1-dimensional subspace spanned by $\psi$. I.E.

$$P_\psi(\phi) = \langle \psi, \phi \rangle \psi \quad \text{for all } \phi \in \mathcal{H}.$$

To see this: use $\psi$ to generate an ONB. calculate the trace ...

Note: In general, $P : \mathcal{H} \to \mathcal{H}$ is an orthogonal projection if $P^2 = P$ and $P^* = P$.