Lecture 22

Infinite systems and the GNS Representation

The GNS Construction

Let $\mathcal{A}$ be a unital C^*-algebra.

Ex. Often $\mathcal{A} = \mathcal{Q}_\Gamma$ where (Γ, d) is a regular metric space.

Recall:

A representation of $\mathcal{A}$ on a Hilbert Space $\mathcal{H}$ is a linear map $\pi : \mathcal{A} \to B(\mathcal{H})$ for which

i) $\pi(\mathbb{1}) = \mathbb{1}$

ii) $\pi(A^*) = \pi(A)^*$ for all $A \in \mathcal{A}$

iii) $\pi(AB) = \pi(A) \cdot \pi(B)$ for all $A, B \in \mathcal{A}$.

A vector $\Omega \in \mathcal{H}$ is said to be cyclic for a representation π if

$$\mathcal{D}_\Omega = \{ \pi(A)\Omega : A \in \mathcal{A} \} \subset \mathcal{H}$$

is a dense subspace of $\mathcal{H}$. A representation π is said to be cyclic if there is a cyclic vector for it.

<u>Theorem</u> (GNS Construction)

Let ω be a state on a unital C^* algebra $\mathfrak{A}$.
Then, there exists a Hilbert space $\mathcal{H}_\omega$, a representation π_ω of $\mathfrak{A}$ on $\mathcal{H}_\omega$, and a vector $\Omega_\omega \in \mathcal{H}_\omega$ that is cyclic for π_ω and also satisfies:

$$\omega(A) = \langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle \qquad \text{for all } A \in \mathfrak{A}.$$

Moreover, up to unitary equivalence, the triple $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ is uniquely determined by ω. In other words, if there are two such cyclic representations $(\mathcal{H}_1, \pi_1, \Omega_1)$ and $(\mathcal{H}_2, \pi_2, \Omega_2)$ corresponding to the same state ω on $\mathfrak{A}$, then there is a unitary map $U: \mathcal{H}_1 \to \mathcal{H}_2$ for which

$$\Omega_2 = U\Omega_1 \quad \text{and} \quad \pi_2(A) = U\pi_1(A)U^* \quad \text{for all } A \in \mathfrak{A}$$

In fact, the unitary with these properties is unique.

This theorem illustrates a deep connection between representations and states on a C^*-algebra.
The name comes from two groups who proved this independently:

Israel Gelfand and Mark Naimark

and

Irving Segal

$\Rightarrow$ Hence <u>GNS</u>.

Recall:

We proved some important facts about states on a C^*-algebra.

i) (Cauchy-Schwarz)

If ω is a state on a C^* algebra, then

$$|\omega(A^* B)|^2 \leq \omega(A^* A)\cdot\omega(B^* B) \quad \text{for all } A, B \in \mathfrak{a}.$$

and $\omega(A^* B) = \overline{\omega(B^* A)}$ for all $A, B \in \mathfrak{a}$.

ii) A consequence of the above for non-trivial unital C^*-algebras:

$$|\omega(A^* B A)| \leq \omega(A^* A)\cdot\|B\| \quad \text{for all } A, B \in \mathfrak{a}.$$

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proof:

We first construct the Hilbert space $\mathcal{H}_\omega$.

Let us introduce the notation $\psi_A = A$ for any $A \in \mathfrak{a}$.

Consider the following sesquilinear form:

$$\langle \psi_A, \psi_B \rangle := \omega(A^* B) \quad \text{for all } A, B \in \mathfrak{a}.$$

This mapping is 2nd component linear and 1st component conjugate linear.

By Cauchy-Schwarz,

$$\langle \psi_A, \psi_B \rangle = \omega(A^* B) = \overline{\omega(B^* A)} = \langle \psi_B, \psi_A \rangle$$

and so this mapping is conjugate symmetric.

Since a state is a positive linear functional, it is clear that

$$\langle \psi_A, \psi_A \rangle = \omega(A^*A) \geq 0 \qquad \text{for all } A \in \mathfrak{A}.$$

Thus this $\langle \cdot, \cdot \rangle$ is almost an inner product.

It is possible, however, that there are $A \in \mathfrak{A}$ with

$$\omega(A^*A) = 0 \quad \text{but} \quad A \neq 0.$$

To construct the Hilbert space of interest, we must "get rid" of these elements.

Let us denote by

$$\mathcal{I} = \{ A \in \mathfrak{A} : \omega(A^*A) = 0 \}.$$

Claim: $\mathcal{I} \subset \mathfrak{A}$ is a subspace and moreover, if $A \in \mathfrak{A}$ and $B \in \mathcal{I}$, then $AB \in \mathcal{I}$.

In words, the 2nd part of this claim establishes that $\mathcal{I}$ is a left ideal in $\mathfrak{A}$.

Let us 1st check that $I \subset \mathfrak{A}$ is a subspace.

Let $A \in \mathfrak{I}$. Take $\alpha \in \mathbb{C}$ and set $\hat{A} = \alpha A$.

clearly $\omega(\hat{A}^* \hat{A}) = \omega(|\alpha|^2 A^* A) = |\alpha|^2 \omega(A^* A) = 0$.

Let $A, B \in \mathfrak{I}$ and take $\alpha, \beta \in \mathbb{C}$. set $\hat{A} = (\alpha A + \beta B)$

Then

$$\omega(\hat{A}^* \hat{A}) = \omega\left((\bar{\alpha} A^* + \bar{\beta} B^*)(\alpha A + \beta B) \right)$$

$$= |\alpha|^2 \underbrace{\omega(A^* A)}_{=0} + \bar{\alpha}\beta\, \omega(A^* B) + \bar{\beta}\alpha\, \omega(B^* A) + |\beta|^2 \overbrace{\omega(B^* B)}^{=0}$$

$$= \bar{\alpha}\beta\, \omega(A^* B) + \bar{\beta}\alpha\, \omega(B^* A)$$

$$= 0 \quad \text{by Cauchy-Schwarz.}$$

Thus $I \subset A$ is a subspace.

Let $A \in \mathfrak{A}$ and $B \in I$. Then

$$0 \le \omega((AB)^* (AB)) = \omega(B^* A^* A B) \le \underbrace{\omega(B^* B)}_{0} \cdot \|A^* A\|$$

where we used the consequence of Cauchy-Schwarz.

Thus I is a left-ideal as claimed.

We use I to define equivalence classes.

For any $A \in \mathfrak{a}$, set

$$[A] = \{A + A' : A' \in I\} \subset \mathfrak{a}.$$

These are the equivalence classes in $\mathfrak{a}$ defined by I.

The quotient space

$$H_0 = \mathfrak{a}/I = \{[A] : A \in \mathfrak{a}\}$$

is also a complex vector space. (Here we use that $I \subset \mathfrak{a}$ is a subspace.) In fact, arithmetic is defined as follows:

$$\alpha[A] := [\alpha A] \quad \text{for any } \alpha \in \mathbb{C} \text{ and } A \in \mathfrak{a}.$$

$$[A] + [B] := [A+B] \quad \text{for any } A, B \in \mathfrak{a}.$$

We now claim that the previous sesquilinear form is an inner product on this vector space.

<u>Define</u>: $\langle \cdot, \cdot \rangle : H_0 \times H_0 \to \mathbb{C}$ by setting

$$\langle [A], [B] \rangle = \omega(\hat{A}^* \hat{B})$$

where $[A], [B] \in H_0$ and $\hat{A} \in [A]$ and $\hat{B} \in [B]$.

We must check that this is well defined.

Note that $\hat{A} \in [A]$ and $\hat{B} \in [B]$ means that there are $A', B' \in I$ with $\hat{A} = A + A'$ and $\hat{B} = B + B'$

The quantity:

$$\omega((\hat{A})^* \hat{B}) = \omega((A+A')^*(B+B'))$$

$$= \omega(A^*B) + \cancelto{0}{\omega(A^*B')} + \omega((A')^*B) + \cancelto{0}{\omega((A')^*B')}$$

since I is a left ideal

$$= \omega(A^*B) + \cancelto{0}{\omega((A')^*B)}$$

by Cauchy-Schwarz

and thus

$$\omega((\hat{A})^* \hat{B}) = \omega(A^*B)$$

independent of the chosen representative of the equivalence class. Hence

$$\langle [A], [B] \rangle = \omega(A^*B) \qquad \text{is well-defined.}$$

By the previous arguments, this quantity is "almost" an inner product. Note that $0 \in \mathfrak{A}$ and $[0] = I$ by construction. Thus

$$\langle [0], [0] \rangle = \omega(0^*0) = 0.$$

If $[A] \neq [0]$, then $\exists\ \hat{A} \in [A]$ with $\hat{A} \notin I$. In this case,

$$\langle [A], [A] \rangle = \omega(\hat{A}^* \hat{A}) > 0 .$$

This proves that $(H_0, \langle \cdot, \cdot \rangle)$ is a Pre-Hilbert space.

Denote by H_ω the completion of H_0 w.r.t. the norm induced by this inner product. This is the desired Hilbert space.

We now define the representation.

We will define a map $\pi_\omega : \mathfrak{A} \to B(H_\omega)$.

Let $A \in \mathfrak{A}$. Define $\pi_\omega(A)$ on H_0 by setting:

$$\pi_\omega(A)[B] = [AB] \qquad \text{for any } [B] \in H_0.$$

This map is linear as:

$$\begin{aligned} \pi_\omega(A)\big(\alpha[B] + \beta[C]\big) &= \pi_\omega(A)\big([\alpha B + \beta C]\big) \\ &= [A(\alpha B + \beta C)] \\ &= \alpha[AB] + \beta[AC] \\ &= \alpha\, \pi_\omega(A)[B] + \beta\, \pi_\omega(A)[C]. \end{aligned}$$

It is also bounded:

$$\begin{aligned} \|\pi_\omega(A)[B]\|^2 = \|[AB]\|^2 &= \langle [AB], [AB] \rangle \\ &= \omega\big((AB)^*(AB)\big) \\ &= \omega(B^* A^* A B) \le \omega(B^* B) \cdot \|A^* A\| \\ &= \|[B]\|^2 \cdot \|A\|^2 \end{aligned}$$

Thus $\pi_\omega(A)$ defines a bounded linear map on $\mathcal{H}_0$.

By the Bounded linear transformation theorem, $\pi_\omega(A)$ uniquely extends to a map on $\mathcal{H}_\omega$ with

$$\|\pi_\omega(A)\| \leq \|A\| \quad \text{for all } A \in \mathcal{a}.$$

We claim that this linear map is the desired representation.

<u>Check:</u>

i) $\pi_\omega(\mathbb{1}) = \mathbb{1}$.

Note that
$$\pi_\omega(\mathbb{1})[B] = [\mathbb{1}B] = [B] \quad \text{for all } [B] \in \mathcal{H}_0.$$

Thus $\pi_\omega(\mathbb{1})$ acts like the identity on $\mathcal{H}_0$. ✓

ii) $\pi_\omega(A^*) = \pi_\omega(A)^*$. Fix $A \in \mathcal{a}$.

Note that:
$$\begin{aligned}\langle \pi_\omega(A^*)[B], [C] \rangle &= \langle [A^*B], [C] \rangle \\ &= \omega((A^*B)^*C) \\ &= \omega(B^*AC) \\ &= \langle [B], [AC] \rangle \\ &= \langle [B], \pi_\omega(A)[C] \rangle\end{aligned}$$

for all $[B], [C] \in \mathcal{H}_0$. This suffices to prove ii). (Check!)

iii) $\pi_\omega(AB) = \pi_\omega(A)\pi_\omega(B)$ for all $A, B \in \mathfrak{a}$.

Let $A, B \in \mathfrak{a}$. Note that

$$\pi_\omega(AB)[C] = \cancel{\text{that}}\ [ABC] = \pi_\omega(A)[BC]$$
$$= \pi_\omega(A)\pi_\omega(B)[C].$$

Since H_0 is dense in H_ω, iii) is clear.

Let us now construct Ω_ω.

Set

$$\Omega_\omega = [\mathbb{1}] \in H_0 \subset H_\omega.$$

Note that

$$\mathcal{D}_{\Omega_\omega} = \{\pi_\omega(A)\Omega_\omega : A \in \mathfrak{a}\} = \{[A] : A \in \mathfrak{a}\} = H_0$$

and this set is dense by construction of H_ω.

Thus Ω_ω is cyclic for π_ω.

Note further that

$$\langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle = \langle [\mathbb{1}], [A] \rangle$$
$$= \omega(\mathbb{1}^* A)$$
$$= \omega(A).$$

This completes the GNS construction.

On Uniqueness

Suppose $(\mathcal{H}_1, \pi_1, \Omega_1)$ and $(\mathcal{H}_2, \pi_2, \Omega_2)$ are two GNS triples associated to a state ω on $\mathcal{A}$.

By assumption, this means that:

$$\langle \Omega_1, \pi_1(A)\Omega_1 \rangle_1 = \omega(A) = \langle \Omega_2, \pi_2(A)\Omega_2 \rangle_2 \quad \text{for all } A \in \mathcal{A}.$$

Since $\mathcal{D}_{\Omega_1} \subset \mathcal{H}_1$ is dense, let us define

$U : \mathcal{H}_1 \to \mathcal{H}_2$ by setting

$$U(\pi_1(A)\Omega_1) = \pi_2(A)\Omega_2 \quad \text{for all } A \in \mathcal{A}.$$

In this case, U is a well-defined linear map on $\mathcal{D}_{\Omega_1}$.

Note that

$$\begin{aligned}
\| U(\pi_1(A)\Omega_1)\|_2^2 &= \langle U(\pi_1(A)\Omega_1), U(\pi_1(A)\Omega_1) \rangle_2 \\
&= \langle \pi_2(A)\Omega_2, \pi_2(A)\Omega_2 \rangle_2 \\
&= \langle \Omega_2, \pi_2(A)^* \pi_2(A)\Omega_2 \rangle_2 \\
&= \langle \Omega_2, \pi_2(A^*A)\Omega_2 \rangle_2 \\
&= \langle \Omega_1, \pi_1(A^*A)\Omega_1 \rangle_1 \\
&= \| \pi_1(A)\Omega_1 \|_1^2
\end{aligned}$$

Thus U is a bounded linear map on $\mathcal{D}_{\Omega_1}$.

Let us also denote by U the bounded linear extension to all of $\mathcal{H}_1$.

We claim that U is isometric:

$$\langle U\pi_1(A)\Omega_1, U\pi_1(B)\Omega_1\rangle_2 = \langle \pi_2(A)\Omega_2, \pi_2(B)\Omega_2\rangle_2$$

$$= \langle \Omega_2, \pi_2(A^*B)\Omega_2\rangle_2$$

$$= \langle \Omega_1, \pi_1(A^*B)\Omega_1\rangle_1$$

$$= \langle \pi_1(A)\Omega_1, \pi_1(B)\Omega_1\rangle_1$$

for all $A, B \in \mathfrak{A}$.

This suffices to conclude that $U: \mathcal{H}_1 \to \mathcal{H}_2$ is an isometry. In fact, since the dense set $\mathcal{D}_{\Omega_2} \subset \text{Range}(U)$, it follows that U is unitary.

The uniqueness statement for the unitary U is easily checked.

<u>Corollary</u> Let $\mathfrak{A}$ be a unital C^*-algebra, ω a state on $\mathfrak{A}$, and α an automorphism of $\mathfrak{A}$ that satisfies $\omega \circ \alpha = \omega$. Then $\exists!\, U \in B(\mathcal{H}_\omega)$ which is unitary and

$$\pi_\omega(\alpha(A)) = U\pi_\omega(A)U^* \quad \text{for all } A \in \mathfrak{A}.$$

We also have that $U\Omega_\omega = \Omega_\omega$.

Remark:

1) $\omega \circ \alpha = \omega$ is often referred to as "α leaves ω invariant".

2) $\pi_\omega(\alpha(A)) = U \pi_\omega(A) U^*$ is often referred to as "α is implementable in the GNS representation of ω".

Proof:

Let $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ be the GNS triple for ω.

Claim: $(\mathcal{H}_\omega, \pi_\omega \circ \alpha, \Omega_\omega)$ is the GNS triple for $\omega \circ \alpha$.

Note: $\tilde{\omega} = \omega \circ \alpha$ is a state on $\mathfrak{A}$.

It is a linear functional and

$$\tilde{\omega}(A^* A) = \omega(\alpha(A^* A)) = \omega(\alpha(A)^* \alpha(A)) \geq 0 \quad ✓$$

Note: $\pi_\omega \circ \alpha : \mathfrak{A} \to B(\mathcal{H}_\omega)$ is a representation:

i) $(\pi_\omega \circ \alpha)(\mathbb{1}) = \pi_\omega(\alpha(\mathbb{1})) = \pi_\omega(\mathbb{1}) = \mathbb{1}$ ✓

ii) $(\pi_\omega \circ \alpha)(A^*) = \pi_\omega(\alpha(A^*)) = \pi_\omega(\alpha(A)^*) = \pi_\omega(\alpha(A))^* = (\pi_\omega \circ \alpha)(A)^*$ ✓

iii) $(\pi_\omega \circ \alpha)(AB) = \pi_\omega(\alpha(AB)) = \pi_\omega(\alpha(A)\alpha(B)) = \pi_\omega(\alpha(A))\, \pi_\omega(\alpha(B))$
$= (\pi_\omega \circ \alpha)(A)\, (\pi_\omega \circ \alpha)(B)$ ✓

It is also clear (using that α is an automorphism) that

$$\mathcal{D}_{\Omega_\omega} = \{ (\pi_\omega \circ \alpha)(A) : A \in \mathfrak{A} \} \subset \mathcal{H}_\omega \quad \text{is a dense subspace.}$$

Finally note that

$$(\omega \circ \alpha)(A) = \omega(\alpha(A)) = \langle \Omega_\omega, \pi_\omega(\alpha(A)) \Omega_\omega \rangle$$

$$= \langle \Omega_\omega, (\pi_\omega \circ \alpha)(A) \Omega_\omega \rangle \quad \text{for all } A \in \mathfrak{A}.$$

This completes the assertion.

Since $\omega = \omega \circ \alpha$, the previous result shows that there is a unique unitary $U : H_\omega \to H_\omega$ for which

$$U \Omega_\omega = \Omega_\omega \quad \text{and} \quad (\pi_\omega \circ \alpha)(A) = U \pi_\omega(A) U^*$$

for all $A \in \mathfrak{A}$. This is the claimed result.