Lecture 5

On Tensor Products

Any two quantum systems, described by Hilbert Spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, can be considered as a composite system. The Hilbert space for the composite system is given by the tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$. This notion readily extends to any finite collection of Hilbert spaces.

Goals for today:

1) Characterize the tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$.
2) Realize this tensor product in some important examples.

Some background in linear algebra

A metric space, in particular a Hilbert space, is said to be separable if there is a countable dense subset. For this lecture, we will only consider separable, complex Hilbert spaces.

Note:

- All finite dimensional Hilbert spaces are separable.
- If (X, μ) is a σ-finite measure space, then $L^2(X, d\mu)$ is separable.

In particular, $L^2(\mathbb{R}^n, dx)$ is separable for any $n \geq 1$.

<u>Fact</u>: Every separable Hilbert space has a countable orthonormal basis (ONB).

- when H is finite dimensional, this statement is clear.
- when H is infinite dimensional, the notion of an ONB will be reviewed in the notes online.

<u>Briefly</u>: a sequence $\{x_n\}_{n=1}^{\infty}$ in H is said to be an ONB if

i) $\langle x_n, x_m \rangle = \delta_{n,m}$ for all $n,m \geq 1$

(i.e. the sequence is an orthonormal set.)

and

ii) for every $x \in H$,

$$x = \lim_{N \to \infty} \sum_{n=1}^{N} \langle x_n, x \rangle x_n \qquad \left(= \sum_{n=1}^{\infty} \langle x_n, x \rangle x_n \right)$$

Here the limit is in norm.

Two consequences of this are:

- (completeness) If $x \in H$ and $\langle x_n, x \rangle = 0$ for all $n \geq 1$, then $x = 0$.
- (Parseval) For any $x \in H$,

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x_n, x \rangle|^2$$

In particular, the series on the right-hand-side above is finite for all $x \in H$.

Tensor products of Hilbert Spaces

<u>Definition</u> Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be (non-empty) separable, complex Hilbert spaces. A pair $(\mathcal{H}, \otimes)$ is called a <u>tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$</u> if

- $\mathcal{H}$ is a Hilbert space

  and

- $\otimes$ is a bilinear map $\otimes : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}$, $(\phi, \psi) \mapsto \phi \otimes \psi$, for which the following properties hold
  - For all $\phi_1, \phi_2 \in \mathcal{H}_1$ and $\psi_1, \psi_2 \in \mathcal{H}_2$,

$$(*) \quad \langle \phi_1 \otimes \psi_1, \phi_2 \otimes \psi_2 \rangle_{\mathcal{H}} = \langle \phi_1, \phi_2 \rangle_{\mathcal{H}_1} \cdot \langle \psi_1, \psi_2 \rangle_{\mathcal{H}_2} .$$

  - Whenever $\{e_j\}_{j \geq 1}$ is an ONB of $\mathcal{H}_1$ and $\{f_k\}_{k \geq 1}$ is an ONB of $\mathcal{H}_2$, then $\{e_j \otimes f_k\}_{j,k \geq 1}$ is an ONB of $\mathcal{H}$.

<u>Remarks:</u>

1) A map $B : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}$, $(\phi, \psi) \mapsto B(\phi, \psi)$, is said to be bilinear if

- $B(\lambda \phi_1 + \mu \phi_2, \psi) = \lambda B(\phi_1, \psi) + \mu B(\phi_2, \psi)$ $\quad \forall \phi_1, \phi_2 \in \mathcal{H}_1$, $\psi \in \mathcal{H}_2$ and $\lambda, \mu \in \mathbb{C}$.
- $B(\phi, \lambda \psi_1 + \mu \psi_2) = \lambda B(\phi, \psi_1) + \mu B(\phi, \psi_2)$

$\forall \phi \in \mathcal{H}_1$, $\psi_1, \psi_2 \in \mathcal{H}_2$, and $\lambda, \mu \in \mathbb{C}$.

2) Elements of the form $\phi \otimes \psi \in \mathcal{H}$, where $\phi \in \mathcal{H}_1$ and $\psi \in \mathcal{H}_2$, are called simple tensors.

3) If $\mathcal{H}_1$ and $\mathcal{H}_2$ are separable, complex Hilbert spaces and $(\mathcal{H}, \otimes)$ is a pair for which

- $\mathcal{H}$ is a Hilbert space
- $\otimes$ is a bilinear map (as above) for which (*) holds

Then, for any ONB $\{e_j\}_{j \geq 1}$ in $\mathcal{H}_1$ and $\{f_k\}_{k \geq 1}$ in $\mathcal{H}_2$, the collection $\{e_j \otimes f_k\}_{j,k \geq 1}$ is clearly an O.N. set.

In this case, it is not hard to check that

$$\{e_j \otimes f_k\}_{j,k \geq 1} \text{ is an ONB for } \mathcal{H}$$

if and only if

$$\mathcal{H} = \overline{\operatorname{span}\{e_j \otimes f_k\}_{j,k \geq 1}}$$

Proposition 1: Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be separable, complex Hilbert spaces and $(\mathcal{H}, \otimes)$ a corresponding tensor product.

a) For all $\phi \in \mathcal{H}_1$ and $\psi \in \mathcal{H}_2$,

$$\|\phi \otimes \psi\|_{\mathcal{H}} = \|\phi\|_{\mathcal{H}_1} \cdot \|\psi\|_{\mathcal{H}_2}.$$

b) If $\phi_n \to \phi$ in $\mathcal{H}_1$ and $\psi_n \to \psi$ in $\mathcal{H}_2$, then

$$\phi_n \otimes \psi_n \to \phi \otimes \psi \in \mathcal{H}.$$

c) If the property about ONBs holds for one pair of ONBs, then it holds for any pair of ONBs. In particular, we need only check the 2nd condition on $(\mathcal{H}, \otimes)$ for one pair of ONBs.

Proofs

a) Let $\phi \in \mathcal{H}_1$ and $\psi \in \mathcal{H}_2$,

$$\|\phi \otimes \psi\|_{\mathcal{H}}^2 = \langle \phi \otimes \psi, \phi \otimes \psi \rangle_{\mathcal{H}} \overset{(*)}{=} \langle \phi, \phi \rangle_{\mathcal{H}_1} \cdot \langle \psi, \psi \rangle_{\mathcal{H}_2}$$
$$= \|\phi\|_{\mathcal{H}_1}^2 \cdot \|\psi\|_{\mathcal{H}_2}^2 .$$

b) Suppose $\phi_n \to \phi$ in $\mathcal{H}_1$ and $\psi_n \to \psi$ in $\mathcal{H}_2$, then

$$\|\phi \otimes \psi - \phi_n \otimes \psi_n\|_{\mathcal{H}} = \|(\phi - \phi_n) \otimes \psi + \phi_n \otimes (\psi - \psi_n)\|_{\mathcal{H}}$$
$$\overset{a)}{\leq} \|\phi - \phi_n\|_{\mathcal{H}_1} \cdot \|\psi\|_{\mathcal{H}_2} + \|\phi_n\|_{\mathcal{H}_1} \cdot \|\psi - \psi_n\|_{\mathcal{H}_2}$$
$$\to 0 \quad \text{as } n \to \infty.$$

Here we have used bilinearity and the fact that: if $\phi_n \to \phi$ in $\mathcal{H}$, then $\{\|\phi_n\|\}_{n \geq 1}$ is bounded. (Check!)

c) Let $\{e_j\}_{j\geq 1}$ be an ONB for $\mathcal{H}_1$.

Let $\{f_k\}_{k\geq 1}$ be an ONB for $\mathcal{H}_2$.

Suppose $\{e_j \otimes f_k\}_{j,k\geq 1}$ is an ONB for $\mathcal{H}$.

Let $\{\tilde{e}_j\}_{j\geq 1}$ be an ONB for $\mathcal{H}_1$

Let $\{\tilde{f}_k\}_{k\geq 1}$ be an ONB for $\mathcal{H}_2$.

By our remark, we need only show that

$$\mathcal{H} = \overline{\operatorname{span}\{\tilde{e}_j \otimes \tilde{f}_k\}_{j,k\geq 1}}.$$

In fact, the same remark shows that, by assumption, we know that

$$\mathcal{H} = \overline{\operatorname{span}\{e_j \otimes f_k\}_{j,k\geq 1}}.$$

The proof is complete if we then show that:

For each $l,m \geq 1$, $\quad e_l \otimes f_m \in \overline{\operatorname{span}\{\tilde{e}_j \otimes \tilde{f}_k\}_{j,k\geq 1}}$.

(check this.)

Since $\{\tilde{e}_j\}$ and $\{\tilde{f}_k\}$ are ONBs, it is clear that

$$e_l = \sum_{j\geq 1} c_j \tilde{e}_j \quad \text{and} \quad f_m = \sum_{k\geq 1} d_k \tilde{f}_k.$$

Let $N \geq 1$. Clearly

$$\left(\sum_{j=1}^{N} c_j \tilde{e}_j\right) \otimes \left(\sum_{k=1}^{N} d_k \tilde{f}_k\right) \overset{\text{bilinearity}}{=} \sum_{j,k=1}^{N} c_j d_k \, \tilde{e}_j \otimes \tilde{f}_k \in \operatorname{span}\{\tilde{e}_j \otimes \tilde{f}_k\}_{j,k\geq 1}$$

Using again that $\{\tilde{e}_j\}_{j\geq 1}$ and $\{\tilde{f}_k\}_{k\geq 1}$ are ONBs, it is clear that

$$\sum_{j=1}^{N} c_j \tilde{e}_j \to e_\ell \text{ in } H_1 \quad \text{and} \quad \sum_{k=1}^{N} d_k \tilde{f}_k \to f_m \in H_2$$

and thus

$$e_\ell \otimes f_m = \lim_{N\to\infty} \left(\sum_{j=1}^{N} c_j \tilde{e}_j\right) \otimes \left(\sum_{k=1}^{N} d_k \tilde{f}_k\right) \in \overline{\operatorname{Span}\{\tilde{e}_j \otimes \tilde{f}_k\}_{j,k\geq 1}}$$

where we have used part b). This completes the proof.

<u>Proposition 2</u>: Let H_1 and H_2 be separable, complex Hilbert spaces.

i) There exists a tensor product $(H, \otimes)$ of H_1 and H_2.

ii) If $(H, \otimes)$ and $(\tilde{H}, \tilde{\otimes})$ are both tensor products of H_1 and H_2, then there exists an isometric isomorphism $J: H \to \tilde{H}$ such that

$$J(\phi \otimes \psi) = \phi \tilde{\otimes} \psi \quad \text{for all } \phi \in H_1 \text{ and } \psi \in H_2.$$

In this case, we will denote the (essentially) unique tensor product of H_1 and H_2 by

$$H = H_1 \otimes H_2.$$

Remark:

A linear map $J : H_1 \to H_2$ is called an isometric isomorphism if

i) J is bijective (i.e. 1 to 1 and onto)

ii) $\langle J\varphi_1, J\varphi_2 \rangle_{H_2} = \langle \varphi_1, \varphi_2 \rangle_{H_1}$ for all $\varphi_1, \varphi_2 \in H_1$.

Proof:

We only consider the case where H_1 and H_2 are infinite dimensional. All other cases follow similarly.

i) Take

$$H = \{ a : \mathbb{N}^2 \to \mathbb{C} \mid \sum_{n,m \geq 1} |a(n,m)|^2 < \infty \}.$$

(Often $H = \ell^2(\mathbb{N}^2)$.) For homework, we will check that

- One checks that H is a complex vector space under the usual addition and scalar multiplication of functions.
- One checks that

$$\langle a, b \rangle = \sum_{n,m \geq 1} \overline{a(n,m)}\, b(n,m)$$

defines an inner-product on H.

- One checks that $\|a\| = \sqrt{\langle a, a \rangle} = \sqrt{\sum_{n,m} |a(n,m)|^2}$ is a complete norm, and hence H is a Hilbert space.

Note further that

For any $j,k \geq 1$, $e_{j,k}: \mathbb{N}^2 \to \mathbb{R}$ given by

$$e_{j,k}(n,m) = \delta_{j,n} \cdot \delta_{k,m} \quad \text{for all } n,m \geq 1$$

defines an orthonormal collection of vectors in $\mathcal{H}$.

Moreover, $\{e_{j,k}\}_{j,k \geq 1}$ is an ONB of $\mathcal{H}$.

Let us now define the bilinear map $\otimes$.

Fix $\{e_j\}_{j \geq 1}$ an ONB on $\mathcal{H}_1$ and $\{f_k\}_{k \geq 1}$ an ONB on $\mathcal{H}_2$.

For any $\phi \in \mathcal{H}_1$ and $\psi \in \mathcal{H}_2$, write

$$\phi = \sum_{j \geq 1} c_j e_j \quad \text{and} \quad \psi = \sum_{k \geq 1} d_k f_k .$$

Define

$$\phi \otimes \psi = \sum_{j,k \geq 1} c_j d_k e_{j,k} .$$

Claim: $\phi \otimes \psi \in \mathcal{H}$.

Check

$$\|\phi \otimes \psi\|_{\mathcal{H}}^2 = \sum_{j,k \geq 1} |c_j d_k|^2 = \Big(\sum_{j \geq 1} |c_j|^2\Big)\Big(\sum_{k \geq 1} |d_k|^2\Big)$$
$$= \|\phi\|_{\mathcal{H}_1}^2 \cdot \|\psi\|_{\mathcal{H}_2}^2 < \infty .$$

use ONB i.e. Parseval

Thus $\phi \otimes \psi \in \mathcal{H}$.

The fact that $\otimes$ is bilinear follows from uniqueness of the coefficients in the expansion with respect to an ONB. (check ...)

To see $(*)$, let $\tilde{\Phi} \in H_1$ and $\tilde{\Psi} \in H_2$.

Expanding as before we see that:

$$\langle \Phi \otimes \Psi, \tilde{\Phi} \otimes \tilde{\Psi} \rangle_H = \sum_{j,k \geq 1} \overline{(\Phi \otimes \Psi)(j,k)} \cdot (\tilde{\Phi} \otimes \tilde{\Psi})(j,k)$$

$$= \sum_{j,k \geq 1} \overline{c_j d_k} \cdot \tilde{c}_j \tilde{d}_k$$

$$= \left(\sum_{j \geq 1} \overline{c_j} \tilde{c}_j \right) \left(\sum_{k \geq 1} \overline{d_k} \tilde{d}_k \right)$$

$$= \langle \Phi, \tilde{\Phi} \rangle_{H_1} \cdot \langle \Psi, \tilde{\Psi} \rangle_{H_2}$$

To check the fact about ONBs, note that

$$e_j \otimes f_k = e_{jk}$$

Thus for this choice of bases ($\{e_j\}_{j \geq 1}$ in H_1 and $\{f_k\}_{k \geq 1}$ in H_2)

$$\{e_j \otimes f_k\}_{j,k \geq 1} = \{e_{j,k}\}_{j,k \geq 1}$$ which we know

is an ONB of $H = \ell^2(\mathbb{N}^2)$.

By Proposition 1c), we are done.

To see Proposition 2 ii), let $\{e_j\}_{j\geq 1}$ and $\{f_k\}_{k\geq 1}$ be ONBs in H_1 and H_2 respectively.

Define a linear map $J: H \to \tilde{H}$ by setting

$$J(e_j \otimes f_k) = e_j \tilde{\otimes} f_k \qquad \text{for all } j,k\geq 1.$$

Claim: J is an isometric isomorphism of H into $\tilde{H}$. (Check!).

Example 1 Let $m\geq 1$ and $n\geq 1$ be integers. Let $H_1 = \mathbb{C}^m$ and $H_2 = \mathbb{C}^n$ equipped with the standard inner products.

Claim: $H_1 \otimes H_2 = \mathbb{C}^{m\times n}$

equipped with the Hilbert-Schmidt inner-product.

Recall: If $A, B \in \mathbb{C}^{m\times n}$, then

$$\langle A, B\rangle_{HS} = \mathrm{Tr}[A^* B] = \sum_{j=1}^{n} (A^*B)_{jj} = \sum_{j=1}^{n}\sum_{k=1}^{m} a^*_{jk} b_{kj}$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{m} \overline{a_{kj}}\, b_{kj}$$

Regarding vectors $u \in \mathbb{C}^m$ and $v \in \mathbb{C}^n$ as columns, we define $\otimes : \mathbb{C}^m \times \mathbb{C}^n \to \mathbb{C}^{m \times n}$ by setting:

$$u \otimes v = u v^t = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} (v_1, \dots, v_n)$$

$$= \begin{pmatrix} u_1 v_1 & u_1 v_2 & \cdots & u_1 v_n \\ u_2 v_1 & u_2 v_2 & \cdots & u_2 v_n \\ \vdots & \vdots & & \vdots \\ u_m v_1 & u_m v_2 & \cdots & u_m v_n \end{pmatrix} \in \mathbb{C}^{m \times n}.$$

Bilinearity of $\otimes$ follows from basic matrix arithmetic. Moreover,

$$\langle u \otimes v, \tilde{u} \otimes \tilde{v} \rangle_{\mathbb{C}^{m \times n}} = \sum_{j=1}^{n} \sum_{k=1}^{m} \overline{(u \otimes v)(k,j)} \, (\tilde{u} \otimes \tilde{v})(k,j)$$

$$= \sum_{j,k} \overline{u_k} \, \overline{v_j} \, \tilde{u}_k \tilde{v}_j$$

$$= \left(\sum_{k=1}^{m} \overline{u_k} \tilde{u}_k \right) \left(\sum_{j=1}^{n} \overline{v_j} \tilde{v}_j \right)$$

$$= \langle u, \tilde{u} \rangle_{\mathbb{C}^m} \cdot \langle v, \tilde{v} \rangle_{\mathbb{C}^n}.$$

Finally, if we let $\{e_j\}_{j=1}^m$ and $\{f_k\}_{k=1}^n$ be the standard basis vectors in $\mathbb{C}^m$ and $\mathbb{C}^n$ respectively, then one readily checks that

$$e_j \otimes f_k = e_j f_k^t = E_{jk}$$

where E_{jk} is the matrix with 1 in the (j,k) entry and 0 everywhere else. It is clear that

$\{E_{jk}\}_{j,k \geq 1}$ is an ONB of $\mathbb{C}^{m \times n}$.

Example 2

Let (X, μ) and (Y, ν) be σ-finite measure spaces.

It is known that

$H_1 = L^2(X, d\mu)$ and $H_2 = L^2(Y, d\nu)$ are separable, complex Hilbert spaces.

Claim: $H_1 \otimes H_2 = L^2(X \times Y, d(\mu \times \nu))$.

To see this, let $\phi \in L^2(X, d\mu)$ and $\psi \in L^2(Y, d\nu)$.

Define

$$(\phi \otimes \psi)(x,y) = \phi(x) \cdot \psi(y).$$

All relevant properties can be checked.