Lecture 8

Some more basics from linear algebra

Let $V \neq \{0\}$ be a finite dimensional complex vector space.

Let $e = \{e_1, e_2, \dots, e_n\} \subset V$ be a basis.

For each $v \in V$,

$$v = \sum_{j=1}^{n} c_j e_j$$ and the numbers c_j for $1 \leq j \leq n$ are uniquely determined.

The map $[\cdot]_e : V \to \mathbb{C}^n$ with $v \mapsto [v]_e$ given by

$$[v]_e = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

is an isomorphism of V and $\mathbb{C}^n$. (The map is linear and bijective.)

For each $v \in V$, we say that the column vector $[v]_e$ represents v.

Of course, if you change the basis, you change the way you represent v.

If, in addition, V is an inner-product space and $e = \{e_1, e_2, \dots, e_n\}$ is an orthonormal basis, then

$$[v]_e = \begin{pmatrix} \langle e_1, v \rangle \\ \langle e_2, v \rangle \\ \vdots \\ \langle e_n, v \rangle \end{pmatrix}$$ i.e. we can calculate these coefficients.

If V and W are both finite dimensional complex vector spaces, then we say that a map $T: V \to W$ is a linear transformation if

$$T(u+v) = T(u) + T(v) \qquad \text{forall } u,v \in V$$

and

$$T(\lambda u) = \lambda T(u) \qquad \text{forall } u \in V \text{ and } \lambda \in \mathbb{C}.$$

We denote by $\mathcal{L}(V,W)$ the collection of all linear transformations from V to W.

In the special case that $V=W$, we write

$$\mathcal{L}(V) = \mathcal{L}(V,V) \qquad \text{as short hand.}$$

A linear transformation $T: V \to V$ is often called a linear operator.

Now fix V and W as above.

Let $e = \{e_1, e_2, \dots, e_n\} \subset V$ be a basis and

let $f = \{f_1, f_2, \dots, f_m\} \subset W$ be a basis.

Then for any $T \in \mathcal{L}(V,W)$ and each $1 \le j \le n$

$$T(e_j) = \sum_{k=1}^{m} a_{kj} f_k$$

The matrix with these coefficients (inserted as columns) is called the matrix representing T in this choice of bases:

$$
[T]_{e,f} = \begin{array}{c} \\ f_1 \\ f_2 \\ \vdots \\ f_m \end{array}
\begin{array}{c} \begin{array}{cccc} e_1 & e_2 & \cdots & e_n \end{array} \\ \left(\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{array} \right) \end{array} \in \mathbb{C}^{m \times n}
$$

$\leftarrow$ domain V

range W

The map $[\cdot]_{ef} : \mathcal{L}(V,W) \to \mathbb{C}^{m\times n}$

is also an isomorphism.

Thus any linear transformation can be represented as a matrix (this requires a choice of bases) and each matrix corresponds to a linear transformation.

Again, for $T \in \mathcal{L}(V,W)$ fixed, different choices of bases produce different matrices...

If, in addition, W is an inner product space and $f = \{f_1, f_2, \ldots, f_m\}$ is an orthonormal basis, then

$$T(e_j) = \sum_{k=1}^{m} \langle f_k, T(e_j)\rangle f_k \Rightarrow [T]_{e,f} = \left(\langle f_i, T(e_j)\rangle \right)_{i,j \geq 1}$$

In the special case of $T \in \mathcal{L}(V)$, we often just choose one basis and use it for both the domain and range of T. We usually write

$$[T]_e = [T]_{e,e}$$

and if V is an inner product space and e is orthonormal then

$$[T]_e = \left(\langle e_i, T(e_j) \rangle \right)_{i,j \geq 1}.$$

Given this understanding, most properties of linear transformations can be determined by matrix arithmetic.

Ex 1 Let $e = \{e_1, e_2, \dots, e_n\} \subset V$ be a basis and $f = \{f_1, f_2, \dots, f_m\} \subset W$ be a basis. For any $T \in \mathcal{L}(V,W)$ and each $v \in V$ one checks that

$$[Tv]_f = [T]_{e,f} \cdot [v]_e$$

as matrix arithmetic.

Ex 2. Let U, V, and W be complex vector spaces. Let e be a basis for U, f be a basis for V, and g be a basis for W. For any $S \in \mathcal{L}(U,V)$ and $T \in \mathcal{L}(V,W)$, then $TS \in \mathcal{L}(U,W)$ satisfies

$$[TS]_{e,g} = [T]_{f,g} \, [S]_{e,f} .$$

In linear algebra, many properties of linear maps are described in terms of matrix calculations. One often needs to check that these properties are independent of the matrix representing the linear map.

<u>Example</u> Trace.

Let $A \in M_n$ i.e. $A = (a_{ij})$ is an $n \times n$ matrix with complex coefficients.

$$\mathrm{Tr}[A] = \sum_{j=1}^{n} a_{jj} = \sum_{j=1}^{n} \langle e_j, Ae_j \rangle$$

where by $\{e_j\}_{j=1}^{n}$ we denote the standard basis in $\mathbb{C}^n$.

If $\{f_k\}_{k=1}^{n}$ is any other orthonormal basis in $\mathbb{C}^n$, then

$$e_j = \sum_{k=1}^{n} \langle f_k, e_j \rangle f_k$$

$$\Rightarrow \mathrm{Tr}[A] = \sum_{j=1}^{n} \langle e_j, Ae_j \rangle = \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{k'=1}^{n} \overline{\langle f_k, e_j \rangle} \langle f_{k'}, e_j \rangle \langle f_k, A f_{k'} \rangle$$

$$= \sum_{k=1}^{n} \sum_{k'=1}^{n} \Big(\underbrace{\sum_{j=1}^{n} \langle f_{k'}, e_j \rangle \langle e_j, f_k \rangle}_{= \langle f_{k'}, f_k \rangle = \delta_{k',k}} \Big) \langle f_k, A f_{k'} \rangle$$

$$= \sum_{k=1}^{n} \langle f_k, A f_k \rangle$$

This shows that the trace may be calculated with respect to any orthonormal basis in $\mathbb{C}^n$.

More generally, if $T \in \mathcal{L}(V)$ where V is a finite dimensional complex inner product space, then

$$\mathrm{Tr}[T] = \mathrm{Tr}\big[[T]_e\big]$$

where e is any orthonormal basis of V.

Another useful property of the trace is "cyclicity".

Let $A, B \in M_n$. Then

$$\begin{aligned}
\mathrm{Tr}[AB] = \sum_{j=1}^{n} (AB)_{jj} &= \sum_{j=1}^{n} \sum_{k=1}^{n} a_{jk} b_{kj} \\
&= \sum_{k=1}^{n} \sum_{j=1}^{n} b_{kj} a_{jk} \\
&= \sum_{k=1}^{n} (BA)_{k,k} \\
&= \mathrm{Tr}[BA].
\end{aligned}$$

Let us now use these matrix representations to better understand the tensor product of matrices.

Let $m, n \geq 1$ be integers.

Let $e = \{e_j\}_{j=1}^{m}$ be an orthonormal basis of $\mathbb{C}^m$.

Let $f = \{f_k\}_{k=1}^{n}$ be an orthonormal basis of $\mathbb{C}^n$.

Let $A \in M_m$ and $B \in M_n$.

We have defined

$$A \otimes B \in B(\mathbb{C}^m \otimes \mathbb{C}^n) = \mathcal{L}(\mathbb{C}^m \otimes \mathbb{C}^n)$$

a finite dimensional vector space.

The collection $g = \{e_j \otimes f_k\}_{j,k \geq 1}$ is clearly an orthonormal basis for $\mathbb{C}^m \otimes \mathbb{C}^n$.

Let us represent $A \otimes B$ in this basis.

To do so, we make a choice in organizing this basis:

Write g as:

$$g = \{ e_1 \otimes f_1, e_2 \otimes f_1, \dots, e_m \otimes f_1, e_1 \otimes f_2, e_2 \otimes f_2, \dots, e_m \otimes f_2, \dots, e_m \otimes f_n \}$$

i.e. list the vectors in g as n groups of m elements.

With respect to this ordering:

$$
[A\otimes B]_g = \begin{array}{c|c} & e_1\otimes f_1 \quad e_2\otimes f_1 \quad \cdots \\ \begin{array}{c} e_1\otimes f_1 \\ e_2\otimes f_1 \\ \vdots \end{array} & \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & & & \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix} \end{array}
$$

i.e. $[A\otimes B]_g$ is written as an $n\times n$ block matrix with each block $C_{ij}\in M_m$ for $1\le i,j\le n$.

By construction, the entries in each block can be calculated.

Fix $1\le i,j\le n$. For any $1\le k,l\le m$

$$
\begin{aligned}
((C_{ij}))_{k,l} &= \langle e_k\otimes f_i, (A\otimes B)(e_l\otimes f_j)\rangle \\
&= \langle e_k\otimes f_i, (Ae_l)\otimes(Bf_j)\rangle \\
&= \langle e_k, Ae_l\rangle \underbrace{\langle f_i, Bf_j\rangle}_{\uparrow \text{constant}}
\end{aligned}
$$

$\Rightarrow C_{ij} = b_{ij}A$ where $b_{ij} = \langle f_i, Bf_j\rangle$

(The ij entry of $[B]_f$.)

We conclude that

$$[A\otimes B]_g = \begin{pmatrix} b_{11}A & b_{12}A & \cdots & b_{1n}A \\ b_{21}A & b_{22}A & \cdots & b_{2n}A \\ \vdots & \vdots & & \\ b_{n1}A & b_{n2}A & \cdots & b_{nn}A \end{pmatrix} \in \mathbb{C}^{mn\times mn}$$

Recall the defining property of $A\otimes B$:

For any $u\in\mathbb{C}^m$ and $v\in\mathbb{C}^n$,

$$(A\otimes B)(u\otimes v) = Au\otimes Bv$$

Here $u\otimes v = uv^t = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}(v_1 \cdots v_n) = \begin{pmatrix} u_1v_1 & u_1v_2 & \cdots & u_1v_n \\ u_2v_1 & u_2v_2 & \cdots & u_2v_n \\ \vdots & \vdots & & \vdots \\ u_mv_1 & u_mv_2 & \cdots & u_mv_n \end{pmatrix} \in \mathbb{C}^{m\times n}$

If e and f are the ~~canonical~~ standard basis vectors in $\mathbb{C}^m$ and $\mathbb{C}^n$ respectively, then

$$u\otimes v = \sum_{j=1}^{m}\sum_{k=1}^{n} u_j v_k \, e_j\otimes f_k$$

$$\Rightarrow [u\otimes v]_g = \begin{pmatrix} v_1 u \\ v_2 u \\ \vdots \\ v_n u \end{pmatrix} \in \mathbb{C}^{mn}$$ i.e. stack the n columns above in one long column vector

one checks that

$$(A\otimes B)(u\otimes v) = Au\otimes Bv = Au\,Bv^t$$

can be understood using matrix arithmetic:

$$[A\otimes B]_g\,[u\otimes v]_g = [AuBv^t]_g \,.$$ check!

Partial Traces.

The next result introduces Partial traces.

Proposition 1 Let $m,n \geq 1$ be integers.

Let $A \in B(\mathbb{C}^m \otimes \mathbb{C}^n) = M_{mn}$.

i) There is a unique linear operator $A_1 \in B(\mathbb{C}^m) = M_m$ defined by

$$(*) \quad \langle \phi, A_1 \psi\rangle_{\mathbb{C}^m} = \sum_{k=1}^{n} \langle \phi\otimes f_k, A(\psi\otimes f_k)\rangle_{\mathbb{C}^m\otimes\mathbb{C}^n}$$

for all $\phi, \psi \in \mathbb{C}^m$. Here $\{f_k\}_{k=1}^n$ is any choice of orthonormal basis on $\mathbb{C}^n$.

ii) There is a unique linear operator $A_2 \in B(\mathbb{C}^n) = M_n$, defined by

$$\langle \tilde{\Phi}, A_2 \tilde{\Psi} \rangle_{\mathbb{C}^n} = \sum_{j=1}^{m} \langle e_j \otimes \tilde{\Phi}, A(e_j \otimes \tilde{\Psi}) \rangle_{\mathbb{C}^m \otimes \mathbb{C}^n}$$

for all $\tilde{\Phi}, \tilde{\Psi} \in \mathbb{C}^n$. Here $\{e_j\}_{j=1}^m$ is any choice of orthonormal basis on $\mathbb{C}^m$.

Once Proposition 1 is proven, we define

$$A_1 = \mathrm{Tr}_{\mathbb{C}^n}[A] \quad \text{and} \quad A_2 = \mathrm{Tr}_{\mathbb{C}^m}[A]$$

as the corresponding partial traces.

- <u>Note</u>: With some more work, these notions apply to certain $A \in B(H_1 \otimes H_2)$.

$$A_1 = \mathrm{Tr}_{H_2}[A] \quad \text{and} \quad A_2 = \mathrm{Tr}_{H_1}[A]$$

are then the partial traces over the second and first factors in the tensor product respectively.

We only prove i).
The proof of ii) is almost identical.

<u>Note</u>: Given A, we need only show that for each choice of $\phi, \psi \in \mathbb{C}^m$, the right-hand-side of $(*)$ is independent of the choice of orthonormal basis $\{f_k\}_{k=1}^n$ of $\mathbb{C}^n$.
In fact, if this is true, then the left-hand-side of $(*)$ is well-defined and this uniquely specifies a linear operator in M_m.

Let $\phi, \psi \in \mathbb{C}^m$. Let $\{e_j\}_{j=1}^n$ be another orthonormal basis of $\mathbb{C}^n$. Then

$$\sum_{k=1}^n \langle \phi \otimes f_k, A(\psi \otimes f_k)\rangle_{\mathbb{C}^m \otimes \mathbb{C}^n} = \sum_{k=1}^n \sum_{j=1}^n \sum_{j'=1}^n \overline{\langle e_j, f_k\rangle}_{\mathbb{C}^n} \langle e_{j'}, f_k\rangle_{\mathbb{C}^n} \langle \phi \otimes e_j, A(\psi \otimes e_{j'})\rangle_{\mathbb{C}^m \otimes \mathbb{C}^n}$$

$$= \sum_{j, j'=1} \underbrace{\left(\sum_{k=1}^n \langle e_{j'}, f_k\rangle_{\mathbb{C}^n} \langle f_k, e_j\rangle_{\mathbb{C}^n} \right)}_{= \langle e_{j'}, e_j\rangle = \delta_{j'j}} \langle \phi \otimes e_j, A(\psi \otimes e_{j'})\rangle_{\mathbb{C}^m \otimes \mathbb{C}^n}$$

$$= \sum_{j=1}^n \langle \phi \otimes e_j, A(\psi \otimes e_j)\rangle_{\mathbb{C}^m \otimes \mathbb{C}^n}$$

as claimed.