Lecture 9

Last class we introduced partial traces.

(Again, we will stick to the case of finite dimensions.)

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be finite dimensional (non-zero) Hilbert spaces.

Let $A \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$.

i) There is a unique linear operator $A_1 \in B(\mathcal{H}_1)$ defined by

$$\langle \phi, A_1 \psi \rangle_{\mathcal{H}_1} = \sum_{k \geq 1} \langle \phi \otimes f_k, A(\psi \otimes f_k) \rangle_{\mathcal{H}_1 \otimes \mathcal{H}_2} \quad \text{for all } \phi, \psi \in \mathcal{H}_1$$

where $\{f_k\}_{k \geq 1}$ is any orthonormal basis of $\mathcal{H}_2$. In words, $A_1 = \mathrm{Tr}_{\mathcal{H}_2}(A)$ is the partial trace of A obtained by "tracing out" the degrees of freedom in $\mathcal{H}_2$.

ii) There is a unique linear operator $A_2 \in B(\mathcal{H}_2)$ defined by

$$\langle \tilde{\phi}, A_2 \tilde{\psi} \rangle_{\mathcal{H}_2} = \sum_{j \geq 1} \langle e_j \otimes \tilde{\phi}, A(e_j \otimes \tilde{\psi}) \rangle_{\mathcal{H}_1 \otimes \mathcal{H}_2} \quad \text{for all } \tilde{\phi}, \tilde{\psi} \in \mathcal{H}_2.$$

where $\{e_j\}_{j \geq 1}$ is any orthonormal basis of $\mathcal{H}_1$. In words, $A_2 = \mathrm{Tr}_{\mathcal{H}_1}(A)$ is the partial trace of A obtained by "tracing out" the degrees of freedom in $\mathcal{H}_1$.

There are different ways to characterize these partial traces.

<u>Recall</u>:

For any Hilbert space $\mathcal{H}$, the dual space of $\mathcal{H}$, which we denote by $\mathcal{H}^*$, is defined by $\mathcal{H}^* = B(\mathcal{H}, \mathbb{C})$. Thus $\mathcal{H}^*$ is the collection of all bounded linear maps $f: \mathcal{H} \to \mathbb{C}$. Such maps are often called bounded linear functionals.

<u>Theorem</u> (Hilbert space version of Riesz Representation Theorem)

Let $\mathcal{H}$ be a Hilbert space.

i) To each $y \in \mathcal{H}$, the map $f_y: \mathcal{H} \to \mathbb{C}$ defined by

$f_y(x) = \langle y, x \rangle$ satisfies $f_y \in \mathcal{H}^*$.

ii) To each $f \in \mathcal{H}^*$, there is a unique $y \in \mathcal{H}$ for which

$f(x) = \langle y, x \rangle$ for all $x \in \mathcal{H}$.

In words, the bounded linear functionals over a Hilbert space can be completely specified by "inner products" with respect to a fixed vector. Given this, any Hilbert space can be identified with its dual. This fact plays an important role in mathematics.

<u>Note</u>: This theorem does <u>not</u> require $\mathcal{H}$ to be finite dimensional.

Again, for convenience, we consider the case of finite dimensions.

<u>Theorem</u>: Let H_1 and H_2 be finite dimensional (non-zero) complex Hilbert spaces. Let $A \in B(H_1 \otimes H_2)$.

i) There is a unique linear map $\tilde{A}_1 \in B(H_1)$ defined by

$$(*) \qquad \mathrm{Tr}[\tilde{A}_1 B] = \mathrm{Tr}[A(B \otimes \mathbb{1})] \qquad \text{for all } B \in B(H_1).$$

trace in $B(H_1)$ trace in $B(H_1 \otimes H_2)$

ii) There is a unique linear map $\tilde{A}_2 \in B(H_2)$ defined by

$$(**) \qquad \mathrm{Tr}[\tilde{A}_2 B] = \mathrm{Tr}[A(\mathbb{1} \otimes B)] \qquad \text{for all } B \in B(H_2).$$

trace in $B(H_2)$ trace in $B(H_1 \otimes H_2)$

iii) $\tilde{A}_1 = A_1 = \mathrm{Tr}_{H_2}(A)$ and $\tilde{A}_2 = A_2 = \mathrm{Tr}_{H_1}(A)$.

<u>proof</u>:

We prove i) and the 1^{st} part of iii).

The rest is done quite similarly.

Since H_1 is finite dimensional (and non-zero), there is an integer $m \geq 1$ for which $H_1 \cong \mathbb{C}^m$.

In this case, $B(H_1) = M_m$ and we have already shown that:

For any $m \geq 1$, $H = M_m$ is a Hilbert space when equipped with the Hilbert-Schmidt inner product.

$$\langle C, D \rangle_{HS} = \underbrace{Tr}_{\text{trace in } M_m}[C^* D] \quad \text{for all } C, D \in M_m.$$

Define $f : M_m \to \mathbb{C}$ by setting.

$$f(B) = \underbrace{Tr}_{\text{trace in } B(H_1 \otimes H_2)}[A(B \otimes \mathbb{1})] \quad \text{for all } B \in M_m.$$

We will show that $f \in M_m^*$.

- One readily checks that f is linear. (check!)
- we need to check that f is bounded, i.e, there is $C < \infty$ and

$$|f(B)| \leq C \cdot \|B\|_{HS} \quad \text{for all } B \in M_m.$$

Note that

$$|f(B)| = |\mathrm{Tr}[A(B\otimes \mathbb{1})]|$$

$$= |\langle A^*, B\otimes \mathbb{1}\rangle_{HS}|$$

Note. This is the Hilbert-Schmidt inner-product on $B(H_1 \otimes H_2)$.

$$\xrightarrow{\text{Cauchy Schwartz}} \leq \|A^*\|_{HS} \cdot \|B\otimes \mathbb{1}\|_{HS}$$

$$\boxed{\text{Homework}} \rightarrow = \underbrace{\|A^*\|_{HS} \cdot \|\mathbb{1}\|_{HS}}_{=: C < \infty} \cdot \|B\|_{HS}$$

Note: $\|\mathbb{1}\|_{HS} = \sqrt{\sum_{j\geq 1} \|\mathbb{1} u_j\|^2} = \sqrt{\dim(H_2)} < \infty.$

This proves that $f \in M_m^*$.

By the Riesz-Representation theorem, there is a unique linear map $\hat{A}_1 \in M_m$ for which:

$$\mathrm{Tr}[A(B\otimes \mathbb{1})] = f(B) = \langle \hat{A}_1, B\rangle_{HS}$$

$$= \mathrm{Tr}[\hat{A}_1^* B] \quad \text{forall } B\in M_m.$$

We take $\tilde{A}_1 = \hat{A}_1^*$.

We now relate this map back to partial traces.

Let $\{e_j\}_{j\geq 1}$ be an orthonormal basis for $\mathcal{H}_1$

Let $\{f_k\}_{k\geq 1}$ be an orthonormal basis for $\mathcal{H}_2$.

For any $B \in \mathcal{B}(\mathcal{H}_1)$, we calculate

$$
\begin{aligned}
\mathrm{Tr}[A(B\otimes \mathbb{1})] &= \sum_{j,k\geq 1} \langle e_j\otimes f_k, A(B\otimes \mathbb{1})\, e_j\otimes f_k\rangle_{\mathcal{H}_1\otimes\mathcal{H}_2} \\
&= \sum_{j\geq 1} \sum_{k\geq 1} \langle e_j\otimes f_k, A\, Be_j\otimes f_k\rangle_{\mathcal{H}_1\otimes\mathcal{H}_2} \\
&= \sum_{j\geq 1} \langle e_j, \mathrm{Tr}_{\mathcal{H}_2}[A]\, Be_j\rangle_{\mathcal{H}_1} \\
&= \mathrm{Tr}\left[\mathrm{Tr}_{\mathcal{H}_2}[A]\, B\right]
\end{aligned}
$$

<u>Thus</u> $\tilde{A}_1 = \mathrm{Tr}_{\mathcal{H}_2}[A]$ as claimed.

Very similar calculations yield the same conclusion about the other partial trace.

Some basic properties of partial traces.

<u>Lemma</u> Let H_1 and H_2 be finite dimensional (non-zero) complex Hilbert spaces.

i) Let $A \in B(H_1 \otimes H_2)$.

- For all $C, D \in B(H_1)$,

$$\mathrm{Tr}_{H_2}\left[(C \otimes \mathbb{1}) A (D \otimes \mathbb{1})\right] = C\, \mathrm{Tr}_{H_2}(A)\, D.$$

- For all $\tilde{C}, \tilde{D} \in B(H_2)$,

$$\mathrm{Tr}_{H_1}\left[(\mathbb{1} \otimes \tilde{C}) A (\mathbb{1} \otimes \tilde{D})\right] = \tilde{C}\, \mathrm{Tr}_{H_1}(A)\, \tilde{D}.$$

ii) Let $A \in B(H_1)$ and $B \in B(H_2)$.

- $\mathrm{Tr}_{H_2}[A \otimes B] = \mathrm{Tr}[B] \cdot A$

- $\mathrm{Tr}_{H_1}[A \otimes B] = \mathrm{Tr}[A] \cdot B.$

<u>Note</u>: As a consequence of ii) above, one often defines "normalized" partial traces.

- If $\mathcal{H}_2$ is finite dimensional, then for any $A \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$,

$$\widetilde{\mathrm{Tr}}_{\mathcal{H}_2}[A] = \frac{1}{\dim(\mathcal{H}_2)} \cdot \mathrm{Tr}_{\mathcal{H}_2}[A].$$

- If $\mathcal{H}_1$ is finite dimensional, then for any $A \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$,

$$\widetilde{\mathrm{Tr}}_{\mathcal{H}_1}[A] = \frac{1}{\dim(\mathcal{H}_1)} \mathrm{Tr}_{\mathcal{H}_1}[A].$$

For these "normalized" partial traces, one has that

- $\widetilde{\mathrm{Tr}}_{\mathcal{H}_2}[A \otimes \mathbb{1}] = A$ for all $A \in B(\mathcal{H}_1)$.
- $\widetilde{\mathrm{Tr}}_{\mathcal{H}_1}[\mathbb{1} \otimes A] = A$ for all $A \in B(\mathcal{H}_2)$.

<u>proof of Lemma</u>

i) Let $\phi, \psi \in \mathcal{H}_1$. Then

$$\langle \phi, \mathrm{Tr}_{\mathcal{H}_2}[(C \otimes \mathbb{1}) A (D \otimes \mathbb{1})] \psi \rangle_{\mathcal{H}_1}$$

$$= \sum_{k \geq 1} \langle \phi \otimes f_k, (C \otimes \mathbb{1}) A (D \otimes \mathbb{1}) \psi \otimes f_k \rangle_{\mathcal{H}_1 \otimes \mathcal{H}_2}$$

$$= \sum_{k \geq 1} \langle (C \otimes \mathbb{1})^* \phi \otimes f_k, A\, D\psi \otimes f_k \rangle_{\mathcal{H}_1 \otimes \mathcal{H}_2}$$

(4.)

$$\langle \phi, \mathrm{Tr}_{\mathcal{H}_2}[(C\otimes \mathbb{1})A(D\otimes \mathbb{1})]\psi\rangle_{\mathcal{H}_1} = \sum_{k\geq 1} \langle C^*\phi\otimes f_k, A\, D\psi\otimes f_k\rangle_{\mathcal{H}_1\otimes\mathcal{H}_2}$$

$$= \langle C^*\phi, \mathrm{Tr}_{\mathcal{H}_2}[A]\, D\psi\rangle_{\mathcal{H}_1}$$

$$= \langle \phi, C\,\mathrm{Tr}_{\mathcal{H}_2}[A]\, D\psi\rangle_{\mathcal{H}_1}$$

This proves the 1st claim in i).

The other claim in i) is proven similarly.

ii) Let $\phi, \psi \in \mathcal{H}_1$. Then

$$\langle \phi, \mathrm{Tr}_{\mathcal{H}_2}[A\otimes B]\psi\rangle_{\mathcal{H}_1} = \sum_{k\geq 1} \langle \phi\otimes f_k, (A\otimes B)\psi\otimes f_k\rangle_{\mathcal{H}_1\otimes\mathcal{H}_2}$$

$$= \sum_{k\geq 1} \langle \phi\otimes f_k, A\psi\otimes B f_k\rangle_{\mathcal{H}_1\otimes\mathcal{H}_2}$$

$$= \langle \phi, A\psi\rangle_{\mathcal{H}_1} \cdot \underbrace{\sum_{k\geq 1} \langle f_k, B f_k\rangle_{\mathcal{H}_2}}_{\mathrm{Tr}[B]}$$

This is the 1st claim in ii).

The 2nd claim is proven similarly.

Let us now try to understand these partial traces as matrices.

Let H_1 and H_2 be finite dimensional (non-zero) complex Hilbert spaces.

Let $e = \{e_j\}_{j=1}^m$ be an orthonormal basis for H_1.

Let $f = \{f_k\}_{k=1}^n$ be an orthonormal basis for H_2.

Let $g = \{e_j \otimes f_k\}_{j,k \geq 1}$ be an orthonormal basis for $H_1 \otimes H_2$.

List g as in the previous class:

$g = \{e_1 \otimes f_1, e_2 \otimes f_1, \cdots, e_m \otimes f_1, e_1 \otimes f_2, e_2 \otimes f_2, \ldots, e_m \otimes f_2, \ldots, e_m \otimes f_n\}$

as n groups of m vectors.

In this case, for any $A \in B(H_1 \otimes H_2)$,

$$[A]_g = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & & & \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}$$

an $n \times n$ block matrix with $A_{ij} \in B(H_1) = M_m$

for all $1 \leq i,j \leq n$.

By construction:

For each $1 \leq i, j \leq n$ fixed:

$$(A_{ij})_{\ell,\ell'} = \langle e_\ell \otimes f_i, A\, e_{\ell'} \otimes f_j \rangle_{H_1 \otimes H_2} \qquad 1 \leq \ell, \ell' \leq m$$

are the entries of these blocks.

Now, we know that

$$\mathrm{Tr}_{H_2}[A] \in B(H_1) = M_m$$

Let us ask: What is the matrix

$$[\mathrm{Tr}_{H_2}[A]]_e \;.$$

To see this, we calculate its entries:

Fix $1 \leq j, j' \leq m$. Then

$$\langle e_j, \mathrm{Tr}_{H_2}[A]\, e_{j'} \rangle_{H_1} = \sum_{k=1}^{n} \langle e_j \otimes f_k, A\, e_{j'} \otimes f_k \rangle_{H_1 \otimes H_2}$$

$$= \sum_{k=1}^{n} (A_{kk})_{j,j'}$$

and thus

$$[\mathrm{Tr}_{H_2}[A]]_e = \sum_{k=1}^{n} A_{kk} \qquad \left(\text{the sum of the diagonal blocks!}\right)$$

We also know that:

$$\mathrm{Tr}_{H_1}[A] \in B(H_2) = M_n .$$

Let us ask: What is the matrix $[\mathrm{Tr}_{H_1}[A]]_f$?

Fix $1 \le k, k' \le n$ and calculate:

$$\langle f_k, \mathrm{Tr}_{H_1}[A] f_{k'} \rangle_{H_2} = \sum_{j=1}^{m} \langle e_j \otimes f_k, A\, e_j \otimes f_{k'} \rangle_{H_1 \otimes H_2}$$
$$= \sum_{j=1}^{m} (A_{kk'})_{j,j}$$
$$= \mathrm{Tr}[A_{k,k'}]$$

Thus $[\mathrm{Tr}_{H_1}[A]]_f$ is the matrix obtained from $[A]_g$ by taking the trace of each of the n^2 blocks.