XY-Model - one dimensional system of N spin sites, with Nearest-Neighbor spin interactions in the XY plane only and possible external magnetic field in the Z-direction $\mathcal{H}_B \sim -\vec{S}\cdot(B\hat{z}) \sim BS_z$.
(Not considering cyclic BC's)

$$\mathcal{H}_\gamma = \sum_{i=1}^{N-1} \mu_i\left[(1+\gamma_i)S_i^x S_{i+1}^x + (1-\gamma_i)S_i^y S_{i+1}^y\right] + \sum_{i=1}^{N} \nu_i S_i^z$$

where $S_i^x = \frac{1}{2}\sigma_i^x$, $S_i^y = \frac{1}{2}\sigma_i^y$, $S_i^z = \frac{1}{2}\sigma_i^z$ where $\sigma_i^x, \sigma_i^y, \sigma_i^z$ are the pauli matrices
And where γ_i is the Anisotropy parameter, μ_i is the nearest-Neighbor Coupling strength and ν_i is the Coupling strength to the external magnetic field.
Note: $[S^a_i, S^b_j] = i\epsilon^{abc}\delta_{ij}S^c_i$ → different sites commute
$\mathcal{H}_\gamma$ is then the tensor product of N $\mathbb{C}^2$ spin sites, so (Hilbert space) $H = \bigotimes_{i=1}^{N}\mathbb{C}^2 \to 2^N$, $\mathcal{H}_\gamma \in B(H) \sim 2^{2N}$

The goal is to re-write $\mathcal{H}_\gamma$ in terms of raising and lowering operators $a_i^\dagger, a_i$. We observe that the a's do not satisfy canonical fermionic algebra, so another transformation is needed.
First, consider the a's anti-commutator and commutation relations:
Defining: $a_i^\dagger = S_i^x + iS_i^y$ and $a_i = S_i^x - iS_i^y$
Then, $\{a_i, a_i^\dagger\} = 1$, $\{a_i, a_i\} = \{a_i^\dagger, a_i^\dagger\} = 0$
while, $[a_i, a_j] = [a_i, a_j^\dagger] = [a_i^\dagger, a_j^\dagger] = 0$ for $i \neq j$ so $a_i a_j^\dagger = a_j^\dagger a_i$
and $[a_i, a_i^\dagger] = -2S_i^z = -\sigma_i^z$ $\Rightarrow \{a_i, a_j^\dagger\} \neq 0$
The a's do not satisfy fermionic anticommutation relations, so it is convenient to use the Jordan-Wigner Transformation

Jordan-Wigner: Define $\boxed{c_i = \exp\left(\pi i \sum_{j=1}^{i-1} a_j^\dagger a_j\right)a_i \Leftrightarrow c_i^\dagger = a_i^\dagger \exp\left(-\pi i \sum_{j=1}^{i-1} a_j^\dagger a_j\right)}$
Then we can show that the c operators satisfy canonical fermion algebra
$\boxed{\text{CAR}}$: $\{c_i, c_j^\dagger\} = \delta_{ij}$, $\{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0$ (Canonical Anticommutation relations)(CAR)
To show this observe that,

$$c_i^\dagger c_i = a_i^\dagger \exp\left(-\pi i \sum_{j=1}^{i-1} a_j^\dagger a_j\right)\exp\left(\pi i \sum_{j=1}^{i-1} a_j^\dagger a_j\right)a_i = a_i^\dagger a_i$$

To see this, first note that, $(a_i^\dagger a_i)^2 = a_i^\dagger a_i a_i^\dagger a_i = a_i^\dagger(1 - a_i^\dagger a_i)a_i = a_i^\dagger a_i$, $a_i^2 = a_i^{\dagger 2} = 0$

$$\exp(\pm\pi i a_i^\dagger a_i) = \sum_{k=0}^{\infty}\frac{1}{k!}(\pm\pi i)^k (a_i^\dagger a_i)^k = 1 + \left(\sum_{k=1}^{\infty}\frac{1}{k!}(\pm\pi i)^k\right)a_i^\dagger a_i = 1 + (e^{\pm\pi i} - 1)a_i^\dagger a_i$$
$$= 1 - 2a_i^\dagger a_i = 1 - 2(S_i^x + iS_i^y)(S_i^x - iS_i^y) = -2S_i^z = -\sigma_i^z$$

Because a_i's commute with disjoint a_j's, (disjoint sites commute)

$$\exp\left(\pi i \sum_{j=n}^{m} a_j^\dagger a_j\right) = \prod_{j=n}^{m}\exp(\pi i a_j^\dagger a_j) = 1 \otimes 1 \ldots \otimes(-\sigma_n^z)\otimes(-\sigma_{n+1}^z)\otimes\ldots\otimes(-\sigma_m^z)\otimes 1 \ldots$$

denote this as P_{nm}

Then $P_{1\,i-1} = \exp\left(\pi i \sum_{j=1}^{i-1} a_j^\dagger a_j\right) = (-\sigma_1^z)\otimes(-\sigma_2^z)\ldots\otimes(-\sigma_{i-1}^z)\otimes 1 \otimes \ldots$

Then $P_{1\,i-1}^\dagger = \exp\left(-\pi i \sum_{j=1}^{i-1} a_j^\dagger a_j\right) = P_{1\,i-1}$ ⇒ self Adjoint

Then clearly,
$$\exp\left(-\pi i \sum_{j=1}^{i-1} a_j^\dagger a_j\right)\exp(+) = P_{1\,i-1}^\dagger P_{1\,i-1} = 1 \Rightarrow c_i^\dagger c_i = a_i^\dagger a_i$$
as $(\sigma_i^z)^2 = 1$

Anti-Commutator of c_i

$c_i^\dagger c_i = a_i^\dagger a_i \Rightarrow \exp(\pi i \sum_{j=1}^{i-1} a_j^\dagger a_j) = \exp(\pi i \sum_{j=1}^{i-1} c_j^\dagger c_j)$

And so we can define the inverse transformation

$$a_i = \exp(-\pi i \sum_{j=1}^{i-1} c_j^\dagger c_j) c_i \quad , \quad a_i^\dagger = c_i^\dagger \cdot \exp(\pi i \sum_{j=1}^{i-1} c_j^\dagger c_j)$$

or $a_i = P_{1i-1}^\dagger c_i$, $a_i^\dagger = c_i^\dagger P_{1i-1}$, $c_i = P_{1i-1} a_i$, $c_i^\dagger = a_i^\dagger P_{1i-1}$

Observe that, $\{\sigma_i^z, a_i\} = 0$ and $[\sigma_i^z, a_j] = 0$ for $i \neq j$ disjoint operators

This implies,

$$\{P_{mn}, a_j\} = 0 \quad \text{for} \quad m \leq j \leq n$$

$$[P_{mn}, a_j] = 0 \quad \text{for} \quad j < m \text{ or } j > n \quad \text{disjoint operators } P_{mn}, a_j$$

Now we can compute $\{c_i, c_i^\dagger\} = P_{1i-1} a_i a_i^\dagger P_{1i-1} + a_i^\dagger P_{1i-1} P_{1i-1} a_i$

Since $i \neq i-1$, P_{1i-1} commutes with a_i and $a_i^\dagger$ $\Rightarrow \{c_i, c_i^\dagger\} = \{a_i, a_i^\dagger\} = 1$

Now compute $\{c_i, c_j^\dagger\}$ for $j > i$

$\{c_i, c_j^\dagger\} = P_{1i-1} a_i a_j^\dagger P_{1j-1} + a_j^\dagger P_{1j-1} P_{1i-1} a_i$

The P's and a's are disjoint on the first term $\Rightarrow \{c_i, c_j^\dagger\} = a_i P_{1i-1} P_{1j-1} a_j^\dagger + a_j^\dagger P_{1j-1} P_{1i-1} a_i$

because $j > i$ and P_{1i-1} are tensored σ_i^z's and $(\sigma_i^z)^2 = 1 \Rightarrow P_{1i-1} P_{1j-1} = P_{ij-1}$

$\Rightarrow \{c_i, c_j^\dagger\} = a_i P_{ij-1} a_j^\dagger + a_j^\dagger P_{ij-1} a_i$

And using $\{P_{ij-1}, a_i\} = 0$ and $[P_{ij-1}, a_j^\dagger] = 0$

$$\Rightarrow \{c_i, c_j^\dagger\} = a_i a_j^\dagger P_{ij-1} - a_j^\dagger a_i P_{ij-1} = [a_i, a_j^\dagger] P_{ij-1} = 0 \quad , a_i, a_j^\dagger \text{ disjoint}$$

Similarly for $i > j$

$$\{c_i, c_j^\dagger\} = P_{1i-1} a_i a_j^\dagger P_{1j-1} + a_j^\dagger P_{1j-1} P_{1i-1} a_i = a_i P_{1i-1} P_{1j-1} a_j^\dagger + a_j^\dagger P_{1j-1} P_{1i-1} a_i$$

$$= a_i P_{ji-1} a_j^\dagger + a_j^\dagger P_{ji-1} a_i = [a_j^\dagger, a_i] P_{ji-1} = 0$$

And,

$$\{c_i, c_j\} = P_{1i-1} a_i P_{1j-1} a_j + P_{1j-1} a_j P_{1i-1} a_i = a_i P_{ij-1} a_j + a_j P_{ij-1} a_i \quad , j > i$$

$$= [a_i, a_j] P_{ij-1} = 0$$

And $(c_i)^2 = P_{1i-1} a_i P_{1i-1} a_i = a_i P_{1i-1} P_{1i-1} a_i = a_i^2 = 0$

And similarly $\{c_i^\dagger, c_j^\dagger\} = 0$, $\Rightarrow (c_i)^2 = (a_i)^2 = 0$, $(c_i^\dagger)^2 = 0$, for $i = j$

Therefore, c's satisfy canonical fermion algebra. CAR

$$\{c_i, c_j^\dagger\} = \delta_{ij} \quad , \quad \{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0$$

Re-write Hamiltonian $\mathcal{H}_\gamma$ in terms of c_i operators.

Using def $a_i, a_i^\dagger$ $\Rightarrow$ $S_i^x = \frac{1}{2}(a_i^\dagger + a_i)$, $S_i^y = \frac{1}{2i}(a_i^\dagger - a_i)$, $S_i^z = a_i^\dagger a_i - \frac{1}{2}$,
we can substitute into $\mathcal{H}_\gamma$,

$$\mathcal{H}_\gamma = \sum_{i=1}^{N-1} \mu_i\left((1+\gamma_i) S_i^x S_{i+1}^x + (1-\gamma_i) S_i^y S_{i+1}^y\right)$$

$$= \frac{1}{4}\sum \mu_i\left((1+\gamma_i)(a_i^\dagger + a_i)(a_{i+1}^\dagger + a_{i+1}) - (1-\gamma_i)(a_i^\dagger - a_i)(a_{i+1}^\dagger - a_{i+1})\right)$$

$$= \frac{1}{2}\sum \mu_i\left[a_i^\dagger a_{i+1} + a_i a_{i+1}^\dagger + \gamma_i(a_i^\dagger a_{i+1}^\dagger + a_i a_{i+1})\right]$$

$$= \frac{1}{2}\sum \mu_i\left[a_i^\dagger a_{i+1} + a_{i+1}^\dagger a_i + \gamma_i(a_i^\dagger a_{i+1}^\dagger + a_{i+1} a_i)\right] = \frac{1}{2}\sum \mu_i\left[a_i^\dagger a_{i+1} + \gamma_i a_i^\dagger a_{i+1}^\dagger + h.c.\right]$$

Again, using the fact that disjoint sites commute,
now compute, $a_i^\dagger a_{i+1} = c_i^\dagger P_{i-1} P_i c_{i+1} = c_i^\dagger P_{ii} c_{i+1}$

$P_{ii} = 1 - 2c_i^\dagger c_i \Rightarrow a_i^\dagger a_{i+1} = c_i^\dagger(1 - 2c_i^\dagger c_i)c_{i+1} = c_i^\dagger c_{i+1}$, $(c_i^\dagger)^2 = 0$

$(P_{ii} = 1 - 2a_i^\dagger a_i,\ a_i^\dagger a_i = c_i^\dagger c_i)$

now calculate $a_i^\dagger a_{i+1}^\dagger = c_i^\dagger P_{i-1} c_{i+1}^\dagger P_i = c_i^\dagger P_{ii} c_{i+1}^\dagger = c_i^\dagger(1 - 2c_i^\dagger c_i)c_{i+1}^\dagger = c_i^\dagger c_{i+1}^\dagger$, $c_i^{\dagger 2} = 0$

$\Rightarrow$

$$\mathcal{H}_\gamma = \frac{1}{2}\sum_{i=1}^{N-1} \mu_i\left[c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1} + \gamma_i(c_i^\dagger c_{i+1}^\dagger + c_{i+1} c_i)\right] + \sum_{i=1}^{N} \nu_i(2c_i^\dagger c_i - 1).$$

~~Now we want to diagonalize $\mathcal{H}_\gamma$, that is, Re-write $\mathcal{H}_\gamma = \sum \lambda_k \eta_k^\dagger \eta_k$ + Const. where η_k's are linear combinations of the c_i's~~

~~$\eta_k = \frac{1}{2}\sum_i\left[(\varphi_{ki} + \psi_{ki})c_i + (\varphi_{ki} - \psi_{ki})c_i^\dagger\right]$ where φ_{ki}, ψ_{ki} are real valued.~~

Re-write $\mathcal{H}_\gamma$ as a double sum. For example,

$$\frac{1}{2}\sum_{i=1}^{N-1} \mu_i(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i) = \sum_{ij} c_i^\dagger A_{ij} c_j, \quad \text{then } A_{ij} = \frac{1}{2}(\mu_j \delta_{i,j+1} + \mu_i \delta_{i+1,j})$$

And,

$$\sum \frac{1}{2}\gamma_i(c_i^\dagger c_{i+1}^\dagger + c_{i+1} c_i)\mu_i = \frac{1}{2}\sum(c_i^\dagger B_{ij} c_j^\dagger - c_i B_{ij} c_j), \quad \underline{B_{ij} = \frac{1}{2}(\mu_i\gamma_i\delta_{i+1,j} - \mu_j\gamma_j\delta_{i,j+1})}$$

The external interaction term $\sum_{i=1}^{N} \nu_i(2c_i^\dagger c_i - 1) = \sum \nu_i(c_i^\dagger c_i - c_i c_i^\dagger)$, $(c_i^\dagger c_i = 1 - c_i c_i^\dagger)$

Then, we can modify $\underline{A_{ij} = \frac{1}{2}(\mu_j\delta_{i,j+1} + \mu_i\delta_{i+1,j} + 4\nu_i\delta_{ij})}$

Then,

$$\mathcal{H}_\gamma = \frac{1}{2}\sum_{ij=1}^{N}\left[c_i^\dagger A_{ij} c_j - c_i A_{ij} c_j^\dagger + c_i^\dagger B_{ij} c_j^\dagger - c_i B_{ij} c_j\right]$$

A_{ij} is symmetric real valued and B_{ij} is anti-symmetric.

define $C = (c_1, c_2 \ldots c_N, c_1^\dagger, \ldots c_N^\dagger)^T$ and

$$M = \begin{pmatrix} A & B \\ -B & -A \end{pmatrix}\frac{1}{2}$$

Then,

$C^\dagger M C = \mathcal{H}_\gamma$, M is referred to as the effective Hamiltonian $2N \times 2N$

Diagonalize $\mathcal{H}_\gamma$

define $b = Wc \Rightarrow \mathcal{H}_\gamma = c^\dagger M c = b^\dagger W M W^\dagger b \xrightarrow{} \sum_k \lambda_k b_k^\dagger b_k$ → Construct W to diagonalize M.

so assume $\exists$ linear combination $b_i = g_{ij} c_j + h_{ij} c_j^\dagger$

$\mathcal{H}_\gamma = c^\dagger M c$, where $M = \frac{1}{2}\begin{pmatrix} A & B \\ -B & -A \end{pmatrix}$ → want to diagonalize M with orthog W.

U, V real and orthogonal matrices

we can apply SVD to $A+B$ → $U(A+B)V^T = \Lambda = \mathrm{diag}(\lambda_1, \lambda_2 \dots \lambda_N)$

$\Rightarrow \Lambda = \Lambda^T = V(A-B)U^T$ (Bogoliubov type decomposition)

$\Rightarrow \Lambda^2 = V(A-B)U^T U(A+B)V^T = \boxed{V(A-B)(A+B)V^T} \Rightarrow$ <u>$N\times N$ eigenvalue problem</u>

$R = (A-B)(A+B) \Rightarrow R = R^T \Rightarrow \exists$ orthonormal basis with real eigenvalues that diagonalizes R.

Define $W = \frac{1}{2}\begin{pmatrix} V+U & V-U \\ V-U & V+U \end{pmatrix}$ $\Rightarrow$ W is orthogonal matrix, $W^T W = \mathbb{1}$

then, $W\begin{pmatrix} A & B \\ -B & -A \end{pmatrix} W^T = \frac{1}{4}\begin{pmatrix} (V+U)A - (V-U)B, & (V+U)B - (V-U)A \\ (V-U)A - (V+U)B, & (V-U)B - (V+U)A \end{pmatrix}\begin{pmatrix} V^T + U^T & V^T - U^T \\ V^T - U^T & V^T + U^T \end{pmatrix}$

$$= \frac{1}{4}\begin{pmatrix} (V+U)A(V^T+U^T) - (V-U)B(V^T+U^T) + (V+U)B(V^T-U^T) - (V-U)A(V^T-U^T), & 0 \\ 0, & (V-U)A(V^T-U^T) - (V+U)B(V^T-U^T) + (V-U)B(V^T+U^T) - (V+U)A(V^T+U^T) \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} VAU^T + UAV^T - VBU^T + UBV^T - VBU^T + UBV^T + VAU^T + UAV^T, & 0 \\ 0, & -VAU^T - UAV^T + VBU^T - UBV^T + VBU^T - UBV^T - VAU^T - UAV^T \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 2V(A-B)U^T + 2U(A+B)V^T, & 0 \\ 0, & -2V(A-B)U^T - 2U(A+B)V^T \end{pmatrix} = \begin{pmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{pmatrix}$$

Define the rotated b-operators, $b = Wc \Rightarrow c = W^\dagger b$

Then, $\mathcal{H}_\gamma = c^\dagger M c = b^\dagger W M W^\dagger b = b^\dagger\left[\frac{1}{2}\begin{pmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{pmatrix}\right] b = \frac{1}{2}\sum_{i=1}^{N} \lambda_i (b_i^\dagger b_i - b_i b_i^\dagger)$

(sum over j):

$b_i = \frac{1}{2}(V_{ij} + U_{ij})c_j + \frac{1}{2}(V_{ij} - U_{ij})c_j^\dagger \Rightarrow \{b_i, b_i^\dagger\} = 1 \Rightarrow$ <u>CAR</u>

$\Rightarrow \mathcal{H}_\gamma = \frac{1}{2}\sum \lambda_i (2 b_i^\dagger b_i - \mathbb{1}) = \boxed{\sum \lambda_i b_i^\dagger b_i - E\mathbb{1}} \Rightarrow$ Free Fermion Hamiltonian

<u>Time evolution:</u>

now calculate $[\mathcal{H}_\gamma, b_i] = \lambda_i b_i^\dagger \cancelto{0}{b_i b_i} - b_i \lambda_i b_i^\dagger b_i = -\lambda_i b_i (1 - b_i b_i^\dagger) = -\lambda_i b_i$

$\dot{A} = i[\mathcal{H}, A]$ $\Rightarrow i[\mathcal{H}_\gamma, b_i] = -i\lambda_i b_i \Rightarrow \frac{d}{dt}(\tau_t(b_i)) = i\,\tau_t([\mathcal{H}_\gamma, b_i]) = -i\lambda_i \tau_t(b_i)$, $\tau_0(b_i) = b_i$

$\Rightarrow \tau_t(b_i) = e^{-it\lambda_i} b_i$ → the time evolution of b-operators.

$\tau_t(b_i^\dagger) = \exp(it\lambda_i) b_i^\dagger$

$\Rightarrow \tau_t(b) = \begin{pmatrix} e^{-it\Lambda} & 0 \\ 0 & e^{it\Lambda} \end{pmatrix} b$, but $c = W^T b \rightarrow \tau_t(c) = W^T \tau_t(b)$

Recall that $(WMW^T)^2 = WMW^TWMW^T = WM^2W^T$

$\tau_t(c) = W^T\begin{pmatrix} e^{-it\Lambda} & 0 \\ 0 & e^{it\Lambda} \end{pmatrix} W c = W^T \exp\left(-it\begin{pmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{pmatrix}\right) W c = \exp\left(-it\, W^T\begin{pmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{pmatrix} W\right) c$

But, $WMW^T = \frac{1}{2}\begin{pmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{pmatrix} \Rightarrow \tau_t(c) = \exp(-2itM)\,c$

In general, given $b_i \Rightarrow c = W^\dagger b \Rightarrow a_i = \exp\left(-\pi i \sum_{j=1}^{i-1} c_j^\dagger c_j\right) c_i \Rightarrow \vec{S}_i(t)$

Exact Eigenvalue problem Solution for $\gamma_i = \gamma$, $\mu_i = 1$, $\nu_i = 0$

$$\Rightarrow \quad A = \tfrac{1}{2}(\delta_{i,j+1} + \delta_{i+1,j}) \quad , \quad B = \tfrac{1}{2}\gamma(\delta_{i+1,j} - \delta_{i,j+1})$$

The eigenvalue problem was determined to be $(A-B)(A+B)V_k = \Lambda_k^2 V_k$

Sum over l.

Then, $[(A-B)(A+B)]_{ij} = \frac{1}{4}(\delta_{i,l+1} + \delta_{i+1,l} - \gamma\delta_{i+1,l} + \gamma\delta_{i,l+1})(\delta_{l,j+1} + \delta_{l+1,j} + \gamma\delta_{l+1,j} - \gamma\delta_{l,j+1})$

$$= \frac{1}{4}\big(\delta_{i-1,j+1} + \delta_{ij} + \gamma\delta_{ij} - \gamma\delta_{i-1,j+1} + \delta_{i+1,j+1} + \delta_{i+1,j-1} + \gamma\delta_{i+1,j-1} - \gamma\delta_{i+1,j+1}$$
$$- \gamma\delta_{i+1,j+1} - \gamma\delta_{i+1,j-1} - \gamma^2\delta_{i+1,j-1} + \gamma^2\delta_{i+1,j+1} + \gamma\delta_{i-1,j+1} + \gamma\delta_{ij} + \gamma^2\delta_{ij} - \gamma^2\delta_{i-1,j+1}\big)$$

$$= \frac{1}{4}\big((1-\gamma+\gamma-\gamma^2)\delta_{i-1,j+1} + (1+\gamma+\gamma+\gamma^2)\delta_{ij} + (1-\gamma-\gamma+\gamma^2)\delta_{i+1,j+1} + (1+\gamma-\gamma-\gamma^2)\delta_{i+1,j-1}\big)$$

$$= \boxed{\frac{1-\gamma^2}{4}(\delta_{i-1,j+1} + \delta_{i+1,j-1}) + \frac{1+\gamma^2}{2}\delta_{ij} = R_{ij}} \quad , \quad R_{ij}\psi_{jn} = \Lambda_n^2 \psi_{in} \quad \text{Sum over } j$$

~~The solution is~~ $\Lambda_k = \left(1-(1-\gamma^2)\sin^2\left(\frac{2\pi n}{N}\right)\right)^{1/2}$, $k = \frac{2\pi n}{N}$, $-\frac{N}{2} \le n \le \frac{N}{2}-1$

~~And,~~ $\psi_{jn} = \begin{cases} \sqrt{\frac{2}{N}}\sin\left(\frac{2\pi n}{N}j\right), & n > 0 \\ \sqrt{\frac{2}{N}}\cos\left(\frac{2\pi n}{N}j\right), & n \le 0 \end{cases}$, $\sqrt{\frac{1}{N}}$ ~~norm for~~ $n = 0, -\frac{N}{2}$

__Eigenvalue calculation:__ $N \times N$ System

Consider Discrete Fourier Transform on R_{ij}: $\rightarrow$ Unitary transform

Define $X_k = \frac{1}{\sqrt{N}}\sum_{n=1}^{N} x_n e^{-i2\pi kn/N}$ and inverse transform: $x_n = \frac{1}{\sqrt{N}}\sum_{k=1}^{N} X_k e^{i2\pi kn/N}$

Then, transform R_{kj}: so $\tilde{R} = U^\dagger R U$

$$\tilde{R}_{kl} = \frac{1}{N}\sum_{m,n=1}^{N}\left[\frac{1-\gamma^2}{4}(\delta_{m+1,n-1} + \delta_{m-1,n+1})e^{i2\pi(km-ln)\frac{1}{N}} + \frac{1+\gamma^2}{2}\delta_{mn}e^{i2\pi(km-ln)\frac{1}{N}}\right]$$

$$= \frac{1}{N}\left(\frac{1-\gamma^2}{4}\left(e^{-i2\pi k\frac{2}{N}} + e^{i2\pi k\frac{2}{N}}\right) + \frac{1+\gamma^2}{2}\right)\sum_{n=1}^{N}\left(e^{i2\pi(k-l)\frac{1}{N}}\right)^n$$

And $\sum_{n=1}^{N}\left(e^{i2\pi(k-l)\frac{1}{N}}\right)^n = N\delta_{kl}$

$$\Rightarrow \tilde{R}_{kl} = \delta_{kl}\cdot\left(\cos^2\frac{2\pi k}{N} + \gamma^2\sin^2\frac{2\pi k}{N}\right) = \Lambda_k^2\delta_{kl} \quad \Rightarrow \quad \Lambda_k = \sqrt{1-(1-\gamma^2)\sin^2\left(\frac{2\pi k}{N}\right)}$$

__Eigenvector calculation:__ $\frac{1}{\sqrt{N}}e^{i2\pi kn/N} = \psi_{kn}$ is the nth eigenvector but not real basis.

$$\sum_k R_{jk}\psi_{kn} = \Lambda_n^2\psi_{jn} \Rightarrow \frac{1-\gamma^2}{4}(\psi_{j-2,n} + \psi_{j+2,n}) + \frac{1+\gamma^2}{2}\psi_{jn} = \Lambda_n^2\psi_{jn}$$

$$\psi_{kn} \rightarrow \psi_{kn} + i\varphi_{kn} = \cos\left(\tfrac{2\pi n}{N}k\right) + i\sin\left(\tfrac{2\pi n}{N}k\right)$$

$$\psi_{k\pm 2,n} = \psi_{kn}\psi_{2n} \mp \varphi_{kn}\varphi_{2n} \quad , \quad \varphi_{k\pm 2,n} = \varphi_{kn}\psi_{2n} \pm \psi_{kn}\varphi_{2n}$$

$$\left(\frac{1-\gamma^2}{2}\psi_{2n} + \frac{1+\gamma^2}{2}\right)\psi_{kn} = \Lambda_n^2\psi_{kn}$$

$$\left(\frac{1-\gamma^2}{2}\psi_{2n} + \frac{1+\gamma^2}{2}\right)\varphi_{kn} = \Lambda_n^2\varphi_{kn}$$

Need to construct u and v from linear combinations of ψ_{kn}, φ_{kn}. or $\cos()$, $\sin()$

Eigenvector Calc continued:

$$\text{Let}\quad V_{nK} = \overset{\sqrt{\frac{2}{N}}}{\downarrow}\, Sin\left(\frac{2\pi n}{N}K\right) \quad n>0 \quad , \quad V_{nK} = \overset{\sqrt{\frac{2}{N}}}{\downarrow}\, Cos\left(\frac{2\pi n}{N}K\right) \; , \; n\leq 0 \quad , \quad -\frac{N}{2}\leq n\leq \frac{N}{2}-1$$

$$\Rightarrow \quad U_{nK} = \frac{1}{\Lambda_n}\left(Cos\left(\frac{2\pi n}{N}\right)V_{nK} + \gamma\, Sin\left(\frac{2\pi n}{N}\right)V_{-n,K}\right)$$

$$\text{To show,}\quad U^T\Lambda = (A+B)V^T \Rightarrow U_{nm} = \sum_K \frac{1}{\Lambda_m}(A+B)_{nK}V_{mK}$$

$$U_{mn} = \frac{1}{2\Lambda_m}\sum_K\left(\delta_{n,K+1}+\delta_{n+1,K}+\gamma\,\delta_{n+1,K}-\gamma\,\delta_{n,K+1}\right)V_{mK} = \frac{1}{2\Lambda_m}\left(V_{m,n+1}+V_{m,n-1}-\gamma V_{m,n-1}+\gamma V_{m,n+1}\right)$$

$$V_{m,n\pm1} = Sin\left(\frac{2\pi m}{N}(n\pm1)\right) = V_{mn}\,Cos\left(\frac{2\pi m}{N}\right) \pm V_{-mn}\,Sin\left(\frac{2\pi m}{N}\right)$$

$$U_{mn} = \frac{1}{2\Lambda_m}\left(V_{mn}\,Cos() + \cancel{V_{-mn}\,Sin()} + V_{mn}\,Cos() - \cancel{V_{-mn}\,Sin()} - \gamma\left(\cancel{V_{mn}\,Cos()} - V_{-mn}\,Sin()\right) + \gamma\left(\cancel{V_{mn}\,Cos()} + V_{-mn}\,Sin()\right)\right)$$

$$= \frac{1}{\Lambda_m}\left(V_{mn}\,Cos\left(\frac{2\pi m}{N}\right) + \gamma\, V_{-mn}\,Sin\left(\frac{2\pi m}{N}\right)\right)$$

Notes on pg. 4:

Show that b_i operators are CAR?

$$b_i b_i = \tfrac{1}{4}\big(\overset{g_{ij}}{(v_{ij}+u_{ij})}c_j + \overset{h_{ij}}{(v_{ij}-u_{ij})}c_j^\dagger\big)\big(\overset{g_{ik}}{(v_{ik}+u_{ik})}c_k + \overset{h_{ik}}{(v_{ik}-u_{ik})}c_k^\dagger\big)$$

$$= \tfrac{1}{4}\sum_{j,k}\big(g_{ij}g_{ik}c_jc_k + g_{ij}h_{ik}c_jc_k^\dagger + h_{ij}g_{ik}c_j^\dagger c_k + h_{ij}h_{ik}c_j^\dagger c_k^\dagger\big)$$

$$= \tfrac{1}{4}\sum_{j,k}\big(-g_{ij}g_{ik}c_kc_j + g_{ij}h_{ik}(\delta_{jk} - c_k^\dagger c_j) + h_{ij}g_{ik}(\delta_{jk} - c_kc_j^\dagger) - h_{ij}h_{ik}c_k^\dagger c_j^\dagger\big)$$

$$= -b_ib_i + \tfrac{1}{2}\sum_k h_{ik}g_{ik}, \qquad \sum_k (v_{ik}-u_{ik})(v_{ik}+u_{ik}) = 1-1 = 0$$

$$\Rightarrow (b_i)^2 = 0 \quad \Rightarrow \quad (b_i^\dagger)^2 = 0$$

General Construction of eigenvectors:

Up to a rotation, $\exists$ unique normalized vacuum vector Ω s.t. $b_i\Omega = 0 \ \forall i$.

This is analogous to the QSHO with $\Omega \Rightarrow$ gaussian ground state

We can do this because b_i's satisfy canonical anti-commutation relations CAR

So we can generate any normalized eigenvector from the vacuum state Ω,

$$\Psi_\alpha = (b_1^\dagger)^{\alpha_1}\dots(b_N^\dagger)^{\alpha_N}\Omega, \qquad \alpha = (\alpha_1,\alpha_2 \dots \alpha_N) \in \{0,1\}^N$$

And the Ψ_α's form an ONB on $H = \bigotimes_{j=1}^N \mathbb{C}^2$

with eigenvalues $E_\alpha = \sum_{j:\alpha_j=1}^N \lambda_j - E$, where $E = \frac{1}{2}\sum_{j=1}^N \lambda_j$

Show b_i operators are CAR:

$$\{b_i, b_\ell^\dagger\} = \tfrac{1}{4}\big[(u_{ij}+v_{ij})c_j + (u_{ij}-v_{ij})c_j^\dagger\big]\big[(u_{\ell k}+v_{\ell k})c_k^\dagger + (u_{\ell k}-v_{\ell k})c_k\big]$$
$$+ \tfrac{1}{4}\big[(u_{\ell k}+v_{\ell k})c_k^\dagger + (u_{\ell k}-v_{\ell k})c_k\big]\big[(u_{ij}+v_{ij})c_j + (u_{ij}-v_{ij})c_j^\dagger\big]$$
$$= \tfrac{1}{4}(u_{ij}+v_{ij})(u_{\ell k}+v_{\ell k})\{c_j, c_k^\dagger\} + \tfrac{1}{4}(u_{ij}-v_{ij})(u_{\ell k}-v_{\ell k})\{c_j^\dagger, c_k\}$$
$$+ \tfrac{1}{4}(u_{ij}+v_{ij})(u_{\ell k}-v_{\ell k})\{c_j, c_k\} + \tfrac{1}{4}(u_{ij}-v_{ij})(u_{\ell k}+v_{\ell k})\{c_j^\dagger, c_k^\dagger\}$$
$$= \tfrac{1}{4}\big(u^Tu + v^Tv + u^Tu + v^Tv + (2u^Tv - 2u^Tv)\big)_{i\ell} = \delta_{i\ell}$$

$$\Rightarrow \{b_i, b_j^\dagger\} = \delta_{ij}, \quad \{b_i, b_j\} = \{b_i^\dagger, b_j^\dagger\} = 0 \quad \Rightarrow \quad \underline{\text{CAR}}$$

Canonical anticommutator relations (CAR)