TEAM CONTEST #2

MATHCAMP 2003

All problems are worth the same number of points. No calculators are permitted.

1. Nine points, no three of which are co-linear, are given inside a unit square. Prove that three of them are vertices of a triangle with area less than or equal to $\frac{1}{8}$.

2. Find all triangles $\triangle ABC$ with the following property:
$$(b+c)R = a\sqrt{bc}\,,$$
where the side lengths of $\triangle ABC$ are a, b, c and its circumradius is R.

3. Find all positive integers n such that $n^2 + 3^n$ is a perfect square.

4. Which is larger,
$$\frac{2001^{2003}}{2003^{2001}} \quad \text{or} \quad \frac{2000^{2002}}{2002^{2000}} \quad ?$$

5. For a positive integer n, let
$$f(n) = \frac{4n + \sqrt{4n^2-1}}{\sqrt{2n+1} + \sqrt{2n-1}}\,.$$
Calculate $f(1) + f(2) + \cdots + f(40)$.

6. Let a, b, c be positive real numbers such that $\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ac} = 1$. Prove that
$$\frac{3}{2} \le \frac{ab-1}{ab+1} + \frac{bc-1}{bc+1} + \frac{ca-1}{ca+1} < 2\,.$$

7. If M is a point in the plane of triangle $\triangle ABC$, prove that
$$AM \sin A \le BM \sin B + CM \sin C\,.$$

8. Consider the polynomial $f(x) = x^n + 2x^{n-1} + \cdots + nx + (n+1)$, and let $\epsilon = \cos\frac{2\pi}{n+2} + i\sin\frac{2\pi}{n+2}$. Show that
$$f(\epsilon)f(\epsilon^2)\cdots f(\epsilon^{n+1}) = (n+2)^n\,.$$

Date: Wednesday, July 16, 2003.