# TEAM CONTEST #3

MATHCAMP 2003

All problems are worth the same number of points. No calculators are permitted.

1. In the sequence
$$\frac{1}{1\cdot 2}, \frac{1}{2\cdot 3}, \frac{1}{3\cdot 4}, \ldots, \frac{1}{(n-1)n}, \ldots$$
find all groups of consecutive terms that add up to $\frac{1}{3}$.

2. Let $p \geq 5$ be a prime. Show that the integer $\frac{p^2+2}{3}$ is a sum of three squares.

3. Let $ABCDE$ be a pentagon such that $AB = AE = CD = 1$, $\angle ABC = \angle DEA = 90^\circ$ and $BC + DE = 1$. Find the area of the pentagon.

4. Let
$$r = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{1335}.$$
If $r = a/b$ with $a$ and $b$ relatively prime integers, show that 2003 divides $a$.

5. Solve the equation
$$3^{x^4-5x^2+3} = \frac{x}{x^2+2}.$$

6. Let $\mathbb{N}$ be the set of nonnegative integers.
   Find all functions $f : \mathbb{N} \setminus \{0\} \to M$ such that
$$1 + f(n)f(n+1) = 2n^2(f(n+1) - f(n))$$
for all $n$ when
(a) $M = \mathbb{N}$
(b) $M = \mathbb{Q}$.

7. A positive integer is called *monotonic* if its digits in base 10, read from left to right, are in nondecreasing order. Prove that for each $n \in \mathbb{N}$, there exists an $n$-digit monotonic number which is a perfect square.

8. Prove that if the three angles between pairs of opposite edges of a tetrahedron are all equal, then they are all equal to $90^\circ$.

*Date*: Wednesday, July 23, 2003.