TEAM CONTEST #4

MATHCAMP 2003

All problems are worth the same number of points. No calculators are permitted.

1. Find all sequences of $2n+1$ real numbers $a_1, \ldots, a_{2n+1}$ so that
$$a_1 + a_2 + \cdots + a_{2n+1} = 2n+1$$
and
$$|a_1 - a_2| = |a_2 - a_3| = \cdots = |a_{2n+1} - a_1| .$$
2. Prove that if $a > 1$ is a rational number, then
$$a = \left(1 + \frac{1}{m^2}\right)\left(1 + \frac{1}{m^2+1}\right)\cdots\left(1 + \frac{1}{m^2+n}\right)$$
for some positive integers m and n.
3. Prove that no three points with integer coordinates can be the vertices of an equilateral triangle.
4. Find the point M inside a convex quadrilateral $ABCD$ such that the sum $MA^2 + MB^2 + MC^2 + MD^2$ is miminal.
5. Let a, b, c be positive real numbers. Prove that
$$\frac{a^3}{b^2} + \frac{b^3}{c^2} + \frac{c^3}{a^2} \geq \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} .$$
6. A convex polygon with 1415 sides has a perimeter of 2001 centimetres. Prove that there exist three vertices of this polygon which form a triangle having area less than 1 square centimetre.
7. Let m and n be positive integers such that $\frac{m}{n} < \sqrt{7}$. Prove that
$$\sqrt{7} - \frac{m}{n} > \frac{1}{mn} .$$
8. Let $A \subset \mathbb{C}$ be a finite set with the property that if $z \in A$ then for all positive integers n we have $z^n \in A$.
   (a) Prove that $\sum_{z \in A} z$ is an integer.
   (b) Show that for any $k \in \mathbb{Z}$ we can find a set A as above such that $\sum_{z \in A} z = k$.

Date: Thursday, July 31, 2003.