TEAM CONTEST #5

MATHCAMP 2003

All problems are worth the same number of points. No calculators are permitted.

1. A student writes three two-digit perfect squares on the blackboard, all in a row. She observes that the resulting six-digit number is also a perfect square. Find this number.
2. Let a be an integer. Prove that for any real number x such that $x^2 < 3$, the numbers $\sqrt{3 - x^2}$ and $\sqrt[3]{a - x^3}$ are not both rational.
3. Prove that the sequence
$$2^1 - 1,\ 2^2 - 1,\ \ldots,\ 2^n - 1,\ \ldots$$
contains 2003 consecutive terms which are composite numbers.
4. Let $A_1A_2 \cdots A_n$ be a regular polygon, $n \geq 3$. Find the number of triangles $\triangle A_iA_jA_k$ which are obtuse.
5. Real numbers $x_1, \ldots, x_n$ are chosen from the interval $[2, 4]$ so that
$$x_1 + x_2 + \cdots + x_n = \frac{17n}{6}$$
and
$$x_1^2 + x_2^2 + \cdots + x_n^2 = 9n\,.$$
Prove that n is divisible by 12.
6. Let $ABCD$ be a convex quadrilateral and let E and T be the midpoints of the sides BC and CD respectively. If $AE + AT = 4$, prove that the area of $ABCD$ is less than 8.
7. Prove that any polygon with perimeter 2004 can be covered by a disk of diameter 1002.
8. Find all positive integers a such that $a^n + 2^n + 1$ divides $a^{n+1} + 2^{n+1} + 1$ for some positive integer n.

Date: Wednesday, August 6, 2003.