FIELD THEORY PROBLEM SET 1

MATHCAMP 2003

1. Suppose that the condition $0 \neq 1$ were dropped from the definition of a field. Does this change anything? (That is, does anything become a field that wasn't before?)

2. Let K be a field.
   (a) If x is any element of K, show that $0 \cdot x = x \cdot 0 = 0$.
   (b) If $x_1, \ldots, x_n, y$ are any elements of K, show that $y \cdot (x_1 + \cdots + x_n) = (y \cdot x_1) + \cdots + (y \cdot x_n)$.

   (I suppose I'm obligated to ask a couple of finicky use-the-definitions questions like these, but you should definitely feel free to skip them if they look boring to you.)

3. And now here's a finicky use-the-definitions question about morphisms. If $f : K \to L$ is a morphism of fields and if $x \in K$ is nonzero, show that $f(x)^{-1} = f(x^{-1})$. In fact, show more generally that $f(x^n) = f(x)^n$. (Similarly, you can see that $f(nx) = nf(x)$ for any x in K and any integer n.)

4. Let $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ denote the set of real numbers of the form $a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$ with a, b, c, d rational. Similarly, let $\mathbb{Q}(\sqrt[3]{2})$ denote the set of real numbers of the form $a + b\sqrt[3]{2} + c(\sqrt[3]{2})^2$ with a, b, c rational. Show that $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ and $\mathbb{Q}(\sqrt[3]{2})$ are fields. How many morprhisms $\mathbb{Q}(\sqrt{2}, \sqrt{3}) \to \mathbb{Q}(\sqrt{2}, \sqrt{3})$ are there? How many morphisms $\mathbb{Q}(\sqrt[3]{2}) \to \mathbb{Q}(\sqrt[3]{2})$ are there?

5. **The characteristic of a field.** Let K be a field. Suppose there is a positive integer n such that $n = 0$ in K. Then the smallest positive integer n such that $n = 0$ is called the *characteristic* of K. If no such n exists, then the characteristic of K is said to be 0. (When the characteristic of K isn't 0, we say that K has "positive characteristic".)
   (a) For example, if p is a prime number, show that the characteristic of $\mathbb{F}_p$ is p. Show that the characteristic of $\mathbb{Q}$ is 0.
   (b) If K has positive characteristic, prove that the characteristic of K can't be just any integer you want: it is always a prime number.
   (c) If K has characteristic 0, show that there is exactly one morphism $\mathbb{Q} \to K$. If K has characteristic p, show that there is exactly one morphism $\mathbb{F}_p \to K$.
   (d) If ℓ, p are two distinct primes and K has characteristic ℓ, can there be a morphism $\mathbb{F}_p \to K$? Or $\mathbb{Q} \to K$? If K has characteristic 0 and p is a prime, can there be a morphism $\mathbb{F}_p \to K$?