

Team Contest #1

Mathcamp 2004

1. Determine all pairs of real numbers a and b such that $a + b \in \mathbb{Z}$ and $a^2 + b^2 = 2$.
2. Let $ABCD$ be a square and let E, F be points on AB and AD respectively. Let P be the intersection of EF and AC . Prove that:

$$\frac{1}{AE} + \frac{1}{AF} = \frac{\sqrt{2}}{AP}.$$

3. Let p and q be positive integers such that $1 \leq q \leq p$ and $a = (p + \sqrt{p^2 + q^2})^2$. Prove that a is an irrational number and that its fractional part is greater than 0.75.
4. Let $n \geq 2$ be a positive integer and $x_1, y_1, x_2, y_2, \dots, x_n, y_n$ positive real numbers. Assuming that

$$x_1 + x_2 + \dots + x_n \geq x_1y_1 + x_2y_2 + \dots + x_ny_n$$

prove that

$$x_1 + x_2 + \dots + x_n \leq \frac{x_1}{y_1} + \frac{x_2}{y_2} + \dots + \frac{x_n}{y_n}.$$

5. Let a, b, c be non-zero integers, $a \neq c$, such that

$$\frac{a}{c} = \frac{a^2 + b^2}{b^2 + c^2}$$

Prove that $a^2 + b^2 + c^2$ cannot be a prime number.

6. Find all functions $f: \mathbb{Z} \rightarrow \mathbb{Z}$ with the property that

$$f(m^2 + f(n)) = (f(m))^2 + n$$

for every $m, n \in \mathbb{Z}$.

7. Consider the sequences defined by $a_n = 3n + \sqrt{n^2 - 1}$ and $b_n = 2(\sqrt{n^2 + n} + \sqrt{n^2 - n})$ for all $n = 1, 2, 3, \dots$. Prove that there are integers A, B so that:

$$\sqrt{a_1 - b_1} + \sqrt{a_2 - b_2} + \dots + \sqrt{a_{49} - b_{49}} = A + B\sqrt{2}.$$

8. Let a_1, a_2, a_3, a_4 be the lengths of the sides of a quadrilateral and let s be its semi-perimeter. Prove that:

$$\sum_{k=1}^4 \frac{1}{s + a_k} \leq \frac{2}{9} \sum_{1 \leq i < j \leq 4} \frac{1}{\sqrt{(s - a_i)(s - a_j)}}.$$