Team Contest #2

Mathcamp 2004

1. Real numbers a, b, c satisfy $a^2 + b^2 + c^2 = 3$. Prove that

$$|a| + |b| + |c| - abc \leq 4\,.$$

2. Let D be a point on side BC of a given triangle ABC. The bisectors of angles $\angle ADB$ and $\angle ADC$ intersect AB and AC at M and N respectively, and the bisectors of angles $\angle ABD$ and $\angle ACD$ intersect DM and DN at K and L respectively.

   Prove that $AM = AN$ if and only if MN and KL are parallel.

3. Let n be a positive integer and let a, b, c be real numbers such that $a^n = a + b$, $b^n = b + c$, $c^n = c + a$. Show that $a = b = c$.

4. Let a, b, c be real numbers. Show that

$$\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} = \sqrt[3]{a + b + c}$$

   if and only if $a^3 + b^3 + c^3 = (a + b + c)^3$.

5. Let A be the set $\{3, 6, 10, 15, 21, \ldots, \frac{5000 \cdot 5001}{2}\}$. Show that there exist numbers $a_1, \ldots, a_{2004} \in A$ such that

$$a_1 + \cdots + a_{2003} = a_{2004}\,.$$

6. Let $N = 1!2!3! \cdots 100!$. Prove that one can erase one term from this product to obtain a perfect square; that is, show that $N/k!$ is a perfect square for some $1 \leq k \leq 100$.

7. Find, with proof, the smallest value of $|25^n - 7^m - 3^m|$ for positive integers m and n.

8. Which non-negative integers can be written as a quotient

$$\frac{a^2 + ab + b^2}{ab - 1}$$

   for non-negative integers a, b with $ab \neq 1$?