# Team Contest #3

Mathcamp 2004

1. Find all real numbers $x > 1$ such that $\sqrt[n]{\lfloor x^n \rfloor}$ is an integer for all positive integers $n \geq 2$. (Here $\lfloor a \rfloor$ denotes the greatest integer less than or equal to $a$.)

2. Given a positive integer $n \geq 3$, find the number of arithmetic progressions of length 3 contained in the set $\{1, 2, \ldots, n\}$.

3. Let $a \geq 2$ be an integer. Consider the set

$$A = \left\{ \sqrt{a},\, \sqrt[3]{a},\, \sqrt[4]{a},\, \sqrt[5]{a}, \ldots \right\} .$$

   (a) Show that $A$ does not contain an infinite geometric progression.

   (b) Show that for any $n \geq 3$, the set $A$ contains a geometric progression of length at least $n$.

4. Let $x$, $y$, $z$ be real numbers such that

$$\cos x + \cos y + \cos z = \cos 3x + \cos 3y + \cos 3z = 0 .$$

   Prove that $\cos 2x \cos 2y \cos 2z \leq 0$.

5. Let $\triangle ABC$ be a triangle inscribed in the circle $K$, and consider a point $M$ which lies on the arc from $B$ to $C$ not containing $A$. The tangents from $M$ to the incircle of $\triangle ABC$ intersect $K$ at the points $N$ and $P$. Prove that if $\angle BAC = \angle NMP$, then $\triangle ABC$ and $\triangle MNP$ are congruent.

6. Find all positive real numbers $a$, $b$, $c$ which satisfy the inequalities

$$4(ab + bc + ca) - 1 \geq a^2 + b^2 + c^2 \geq 3(a^3 + b^3 + c^3) .$$

7. Find all increasing functions $f : \{1, 2, \ldots, 10\} \to \{1, 2, \ldots, 100\}$ having the property: $x + y$ is a divisor of $xf(x) + yf(y)$ for all $x, y \in \{1, 2, \ldots, 10\}$. (Note: a function is said to be increasing if $f(x) \leq f(y)$ whenever $x \leq y$.)

8. Prove that any positive rational number can be written in the form

$$\frac{a^3 + b^3}{c^3 + d^3}$$

   where $a$, $b$, $c$, $d$ are positive integers.