Team Contest #4

Mathcamp 2004

1. Let $\{x\}$ denote the fractional part of x.

   (a) Prove that there are infinitely many positive rational numbers x such that $\{x^2\} + \{x\} = 0.99$.

   (b) Prove that there are no positive rational numbers x such that $\{x^2\} + \{x\} = 1$.

2. Find all injective functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(f(n)) \leq \frac{n + f(n)}{2}$$

   for each $n \in \mathbb{N}$. (A function f is injective if $f(x) \neq f(y)$ whenever $x \neq y$.)

3. Let $a < b \leq c < d$ be positive integers so that $ad = bc$ and $\sqrt{d} - \sqrt{a} \leq 1$. Prove that a is a perfect square.

4. Consider the sequence with $a_1 = 3$ and $a_{n+1} = \frac{1}{2}(3a_n^2 + 1) - a_n$ for $n \geq 1$. Show that if n is a power of 3, then n divides a_n.

5. Let $\triangle ABC$ be a right triangle with $\angle A = 90°$, and let D lie on AC such that BD is the angle bisector of $\angle B$. Prove that $BC - BD = 2AB$ if and only if

$$\frac{1}{BD} - \frac{1}{BC} = \frac{1}{2AB} .$$

6. Find all integers n such that $\sqrt{\frac{4n-2}{n+5}}$ is rational.

7. Find the minimum and the maximum of $E(x, y) = \frac{3xy-4x^2}{x^2+y^2}$ where x, y range over the nonzero real numbers.

8. Show that there exist infinitely many systems of positive integers x, y, z, t such that $\text{GCD}(x, y, z, t) = 1$ and

$$x^3 + y^3 + z^2 = t^4 .$$