Team Contest #5

Mathcamp 2004

1. Find all real numbers x which satisfy the equation $3^x + 4^{1/x} = 11$.
2. Let A be a set of real numbers which satisfies
   (a) $1 \in A$;
   (b) if $x \in A$, then $x^2 \in A$;
   (c) if $x^2 - 4x + 4 \in A$ then $x \in A$.

   Show that $2000 + \sqrt{2001} \in A$.
3. Show that if n is a positive integer, then $n^2 + n$ divides $\binom{n^2}{n}$.
4. Determine all primes x and y such that
$$\binom{x}{y} = xy + x + y + 4\,.$$
5. In triangle $\triangle ABC$, the numbers $\tan A$, $\tan B$, and $\tan C$ are all rational. Show that the number $\sin A \cdot \sin B \cdot \sin C$ is rational.
6. Find all functions $f : [0, \infty) \to [0, \infty)$ satisfying the inequalities
$$f(x^2 + x) \le x \le f(x)^2 + f(x)$$
   for all $x \ge 0$.
7. Determine the ordered triples (x, y, z) of positive rational numbers for which $x + \frac{1}{y}$, $y + \frac{1}{z}$, and $z + \frac{1}{x}$ are all integers.
8. Find all polynomials P with real coefficients such that
$$P(x)P(2x^2 - 1) = P(x^2)P(2x - 1)\,.$$