1. Solutions

C99. Find all collections of polynomials \( p_{11}, p_{12}, p_{21}, p_{22} \) with complex coefficients satisfying the relation

\[
\begin{pmatrix}
  p_{11}(XY) & p_{12}(XY) \\
  p_{21}(XY) & p_{22}(XY)
\end{pmatrix}
= \begin{pmatrix}
  p_{11}(X) & p_{12}(X) \\
  p_{21}(X) & p_{22}(X)
\end{pmatrix}
\cdot \begin{pmatrix}
  p_{11}(Y) & p_{12}(Y) \\
  p_{21}(Y) & p_{22}(Y)
\end{pmatrix}.
\]

Solution: We begin with the observation that if

\[ f(XY) = \sum_{i=1}^{n} g_i(X) h_i(Y) \]

then the polynomial \( f \) has at most \( n \) non-zero terms. To see this, first note that for any polynomial in two variables \( q(X,Y) = \sum_{j,k} a_{j,k} X^j Y^k \) we may consider the matrix of coefficients \( (a_{j,k}) \). For the products \( g_i(X) h_i(Y) \), this matrix has rank 1, and so the matrix of coefficients of the polynomial \( \sum_{i=1}^{n} g_i(X) h_i(Y) \) has rank at most \( n \). However, the rank of the matrix associated to \( f(XY) \) is exactly equal to the number of non-zero terms of \( f \), and so this number is at most \( n \).

For example, suppose \( f(XY) = f(X)f(Y) \). Applying our observation, either \( f = 0 \) or \( f \) is a monomial \( f(X) = cX^n \), and in the latter case we see \( c(XY)^n = cX^n cY^n \) and so \( c = 1 \). Hence \( f(X) = 0 \) or \( f(X) = X^n \) for some nonnegative integer \( n \).

Let \( P(X) \) denote our two-by-two matrix of polynomials

\[
\begin{pmatrix}
  p_{11}(X) & p_{12}(X) \\
  p_{21}(X) & p_{22}(X)
\end{pmatrix}.
\]

Then \( P(X) P(Y) = P(XY) \) implies that \( P(1)^2 = P(1) \). This leaves three possibilities for the minimal polynomial for \( P(1) \): the minimal polynomial is either \( P(1) = 0 \), \( P(1)(P(1) - I) = 0 \), or \( P(1) - I = 0 \).
In the middle case, $P(1)$ has eigenvalues 0 and 1, and so there exists a matrix $A$ with complex entries such that

$$P(1) = A^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} A.$$ 

We now consider the three cases in turn.

**Case 1:** If $P(1) = 0$, then $P(X) = P(X)P(1) = 0$. Hence $P$ is identically 0.

**Case 2:** If $P(1)$ satisfies

$$P(1) = A^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} A,$$

we consider the conjugate matrix $Q(X) = AP(X)A^{-1}$. Then it is still the case that $Q(XY) = Q(X)Q(Y)$, and now

$$Q(1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

But then

$$Q(X) = Q(1)Q(X) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} = \begin{pmatrix} q_{11} & q_{12} \\ 0 & 0 \end{pmatrix}$$

and similarly

$$Q(X) = Q(X)Q(1) = \begin{pmatrix} q_{11} & 0 \\ q_{21} & 0 \end{pmatrix}$$

from which it follows that

$$Q(X) = \begin{pmatrix} q_{11} & 0 \\ 0 & 0 \end{pmatrix}$$

and $q_{11}(XY) = q_{11}(X)q_{11}(Y)$. By our earlier observation, it follows that $q_{11}(X) = X^m$ for some nonnegative $m$, and in summary we have obtained:

$$P(X) = A^{-1} \begin{pmatrix} X^m & 0 \\ 0 & 0 \end{pmatrix} A$$

for a complex matrix $A$.

**Case 3:** If we have $P(1) = I$, our solution follows the method of Case 2 but is more complicated. In this case $p_{ij}(XY) = p_{i1}(X)p_{1j}(Y) + p_{i2}(X)p_{2j}(Y)$, and by our initial observation, it follows that each $p_{ij}$ has at most two terms. However, we can say even more: for an invertible complex matrix $A$, the conjugate $Q(X) = A^{-1}P(X)A$ still satisfies
$Q(XY) = Q(X)Q(Y)$, so the entries of $A^{-1}P(X)A$ must also have at most two terms each. Writing down the entries of $A^{-1}P(X)A$ explicitly in terms of the entries of $A$, it is easy to see that among the various terms of $p_{11}, p_{12}, p_{21}, p_{22}$ there can be terms of at most two different degrees—otherwise, it would be possible to arrange a choice of $A$ so that $A^{-1}P(X)A$ had an entry which had three terms.

Each $p_{ij}$ may be therefore written $p_{ij} = a_{ij}X^m + b_{ij}X^n$ for some common pair of distinct integers $m$ and $n$. Using $P(1) = I$, we see moreover that $P(X)$ can be written

$$P(X) = \begin{pmatrix} aX^m + (1-a)X^n & b(X^m - X^n) \\ c(X^m - X^n) & dX^m + (1-d)X^n \end{pmatrix}$$

which we rewrite as

$$P(X) = X^n I + (X^m - X^n)M$$

for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Expanding the condition $P(XY) = P(X)P(Y)$ in terms of the above expression, we quickly obtain

$$(X^m - X^n)(Y^m - Y^n)M^2 = (X^m - X^n)(Y^m - Y^n)M$$

and so $M^2 = M$. As earlier, it follows that either $M = 0$, $M = I$, or $M = A^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}A$. In these three cases we see, respectively, that $P(X) = X^n I$, $P(X) = X^m I$, or

$$P(X) = A^{-1}\left( X^n I + (X^m - X^n) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) A = A^{-1}\begin{pmatrix} X^m & 0 \\ 0 & X^n \end{pmatrix} A.$$ 

In summary, we have shown that $P(X)$ must be of the form:

$$P(X) = 0,$$

$$P(X) = A^{-1}\begin{pmatrix} X^m & 0 \\ 0 & 0 \end{pmatrix} A,$$

or

$$P(X) = A^{-1}\begin{pmatrix} X^m & 0 \\ 0 & X^n \end{pmatrix} A,$$

for some invertible complex matrix $A$.

C100. (Proposed by José Luis Diaz, Universitat Politecnica de Catalunya, Spain.) Let $x_1, x_2, \ldots, x_n$ be positive real numbers, let $S = \sum_{k=1}^n x_k$,
and suppose that \((n - 1)x_k < S\) for all \(k\). Prove that

\[
\prod_{k=1}^{n} (S - (n - 1)x_k) \leq \prod_{k=1}^{n} x_k.
\]

When does equality occur?

**Solution:** (Solved by Michel Bataille, Rouen, France, and David Loeffler, student, Trinity College, Cambridge, UK.)

Observe that

\[
\sum_{k=1}^{n} (S - (n - 1)x_k) = nS - (n - 1)\sum_{k=1}^{n} x_k = S,
\]

and therefore

\[
\sum_{k\neq j} (S - (n - 1)x_k) = S - (S - (n - 1)x_j) = (n - 1)x_j.
\]

Noting that each term in the sum is positive, it follows from the AM-GM inequality that

\[
x_j \geq \prod_{k\neq j} (S - (n - 1)x_k)^{1/(n-1)}.
\]

Multiplying together these inequalities for \(j = 1, \ldots, n\) yields the desired inequality.

To determine when equality holds, note from our use of the AM-GM inequality that we must have \(x_j = S - (n - 1)x_k\) for all \(k \neq j\). This implies that any selection of \(n - 1\) of the \(x_k\)s must all be equal, and so for \(n > 2\), equality holds if and only if the \(x_k\) are all equal. Moreover, one checks easily that equality always holds if \(n = 1\) or \(n = 2\).