CHALLENGE BOARD, VOLUME 27 ISSUE 6

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1. Solutions

C99. Find all collections of polynomials $p_{11}, p_{12}, p_{21}, p_{22}$ with complex coefficients satisfying the relation

$$\begin{pmatrix} p_{11}(XY) & p_{12}(XY) \\ p_{21}(XY) & p_{22}(XY) \end{pmatrix} = \begin{pmatrix} p_{11}(X) & p_{12}(X) \\ p_{21}(X) & p_{22}(X) \end{pmatrix} \cdot \begin{pmatrix} p_{11}(Y) & p_{12}(Y) \\ p_{21}(Y) & p_{22}(Y) \end{pmatrix} .$$

Solution: We begin with the observation that if

$$f(XY) = \sum_{i=1}^{n} g_i(X)h_i(Y)$$

then the polynomial f has at most n non-zero terms. To see this, first note that for any polynomial in two variables $q(X, Y) = \sum_{j,k} a_{j,k}X^jY^k$ we may consider the matrix of coefficients $(a_{j,k})$. For the products $g_i(X)h_i(Y)$, this matrix has rank 1, and so the matrix of coefficients of the polynomial $\sum_{i=1}^{n} g_i(X)h_i(Y)$ has rank at most n. However, the rank of the matrix associated to $f(XY)$ is exactly equal to the number of non-zero terms of f, and so this number is at most n.

For exanple, suppose $f(XY) = f(X)f(Y)$. Applying our observation, either $f = 0$ or f is a monomial $f(X) = cX^n$, and in the latter case we see $c(XY)^n = cX^ncY^n$ and so $c = 1$. Hence $f(X) = 0$ or $f(X) = X^n$ for some nonnegative integer n.

Let $P(X)$ denote our two-by-two matrix of polynomials

$$\begin{pmatrix} p_{11}(X) & p_{12}(X) \\ p_{21}(X) & p_{22}(X) \end{pmatrix} .$$

Then $P(X)P(Y) = P(XY)$ implies that $P(1)^2 = P(1)$. This leaves three possibilities for the minimal polynomial for $P(1)$: the minimal polynomial is either $P(1) = 0$, $P(1)(P(1) - I) = 0$, or $P(1) - I = 0$.

In the middle case, $P(1)$ has eigenvalues 0 and 1, and so there exists a matrix A with complex entries such that

$$P(1) = A^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} A\,.$$

We now consider the three cases in turn.

Case 1: If $P(1) = 0$, then $P(X) = P(X)P(1) = 0$. Hence P is identically 0.

Case 2: If $P(1)$ satisfies

$$P(1) = A^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} A\,,$$

we consider the conjugate matrix $Q(X) = AP(X)A^{-1}$. Then it is still the case that $Q(XY) = Q(X)Q(Y)$, and now

$$Q(1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\,.$$

But then

$$Q(X) = Q(1)Q(X) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} = \begin{pmatrix} q_{11} & q_{12} \\ 0 & 0 \end{pmatrix}$$

and similarly

$$Q(X) = Q(X)Q(1) = \begin{pmatrix} q_{11} & 0 \\ q_{21} & 0 \end{pmatrix}$$

from which it follows that

$$Q(X) = \begin{pmatrix} q_{11} & 0 \\ 0 & 0 \end{pmatrix}$$

and $q_{11}(XY) = q_{11}(X)q_{11}(Y)$. By our earlier observation, it follows that $q_{11}(X) = X^m$ for some nonnegative m, and in summary we have obtained:

$$P(X) = A^{-1} \begin{pmatrix} X^m & 0 \\ 0 & 0 \end{pmatrix} A$$

for a complex matrix A.

Case 3: If we have $P(1) = I$, our solution follows the method of Case 2 but is more complicated. In this case $p_{ij}(XY) = p_{i1}(X)p_{1j}(Y) + p_{i2}(X)p_{2j}(Y)$, and by our initial observation, it follows that each p_{ij} has at most two terms. However, we can say even more: for an invertible complex matrix A, the conjugate $Q(X) = A^{-1}P(X)A$ still satisfies

$Q(XY) = Q(X)Q(Y)$, so the entries of $A^{-1}P(X)A$ must also have at most two terms each. Writing down the entries of $A^{-1}P(X)A$ explicitly in terms of the entries of A, it is easy to see that among the various terms of $p_{11}, p_{12}, p_{21}, p_{22}$ there can be terms of at most two different degrees—otherwise, it would be possible to arrange a choice of A so that $A^{-1}P(X)A$ had an entry which had three terms.

Each p_{ij} may be therefore written $p_{ij} = a_{ij}X^m + b_{ij}X^n$ for some common pair of distinct integers m and n. Using $P(1) = I$, we see moreover that $P(X)$ can be written

$$P(X) = \begin{pmatrix} aX^m + (1-a)X^n & b(X^m - X^n) \\ c(X^m - X^n) & dX^m + (1-d)X^n \end{pmatrix}$$

which we rewrite as

$$P(X) = X^n I + (X^m - X^n)M$$

for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Expanding the condition $P(XY) = P(X)P(Y)$ in terms of the above expression, we quickly obtain

$$(X^m - X^n)(Y^m - Y^n)M^2 = (X^m - X^n)(Y^m - Y^n)M$$

and so $M^2 = M$. As earlier, it follows that either $M = 0$, $M = I$, or $M = A^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}A$. In these three cases we see, respectively, that $P(X) = X^n I$, $P(X) = X^m I$, or

$$P(X) = A^{-1}\left(X^n I + (X^m - X^n)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)A = A^{-1}\begin{pmatrix} X^m & 0 \\ 0 & X^n \end{pmatrix}A\,.$$

In summary, we have shown that $P(X)$ must be of the form:

$$P(X) = 0\,,$$

$$P(X) = A^{-1}\begin{pmatrix} X^m & 0 \\ 0 & 0 \end{pmatrix}A\,,$$

or

$$P(X) = A^{-1}\begin{pmatrix} X^m & 0 \\ 0 & X^n \end{pmatrix}A\,,$$

for some invertible complex matrix A.

C100. (Proposed by José Luis Diaz, Universitat Politecnica de Catalunya, Spain.) Let $x_1, x_2, \ldots, x_n$ be positive real numbers, let $S = \sum_{k=1}^{n} x_k$, and suppose that $(n-1)x_k < S$ for all k. Prove that

$$\prod_{k=1}^{n} (S-(n-1)x_k) \leq \prod_{k=1}^{n} x_k \,.$$

When does equality occur?

Solution: (Solved by Michel Bataille, Rouen, France, and David Loeffler, student, Trinity College, Cambridge, UK.)

Observe that

$$\sum_{k=1}^{n}(S-(n-1)x_k) = nS-(n-1)\sum_{k=1}^{n} x_k = S\,,$$

and therefore

$$\sum_{k\neq j}(S-(n-1)x_k) = S-(S-(n-1)x_j) = (n-1)x_j\,.$$

Noting that each term in the sum is positive, it follows from the AM - GM inequality that

$$x_j \geq \prod_{k\neq j}(S-(n-1)x_k)^{1/(n-1)}\,.$$

Multiplying together these inequalities for $j=1,\ldots,n$ yields the desired inequality.

To determine when equality holds, note from our use of the AM-GM inequality that we must have $x_j = S-(n-1)x_k$ for all $k \neq j$. This implies that any selection of $n-1$ of the x_ks must all be equal, and so for $n>2$, equality holds if and only if the x_k are all equal. Moreover, one checks easily that equality always holds if $n=1$ or $n=2$.