

SERRE WEIGHTS FOR QUATERNION ALGEBRAS

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ABSTRACT. We study the possible weights of an irreducible 2-dimensional mod p representation of $\text{Gal}(\overline{F}/F)$ which is modular in the sense of that it comes from an automorphic form on a definite quaternion algebra with centre F which is ramified at all places dividing p , where F is a totally real field. In most cases we determine the precise list of possible weights; in the remaining cases we determine the possible weights up to a short and explicit list of exceptions.

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1. INTRODUCTION

Let F be a totally real field and let p be a prime number. In this paper we formulate, and largely prove, an analogue of the weight part of Serre's conjecture [Ser87] for automorphic forms on quaternion algebras over F which are ramified at all places dividing p .

In recent years, a great deal of attention has been given to the problem of generalizing the weight part of Serre's conjecture beyond the case of $\text{GL}(2, \mathbb{Q})$, beginning with the seminal paper [BDJ08] which considered the situation for Hilbert modular forms. Let G_F denote the absolute Galois group of F ; then to any irreducible modular representation

$$\overline{\rho} : G_F \rightarrow \text{GL}_2(\overline{\mathbb{F}}_p)$$

there is associated a set of weights $W(\overline{\rho})$, the set of weights in which $\overline{\rho}$ is modular (see section 2 for the definitions of weights and of what it means for $\overline{\rho}$ to be modular of a certain weight). Under the assumption that p is unramified in F , the paper [BDJ08] associated to $\overline{\rho}$ a set of weights $W^?(\overline{\rho})$, and conjectured that $W^?(\overline{\rho}) =$

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$W(\bar{\rho})$. Schein [Sch08] subsequently proposed a generalisation that, in the tame case (where the restrictions of $\bar{\rho}$ to inertia subgroups at places dividing p are semisimple), removes the restriction that p be unramified in F . When p is either unramified or totally ramified in F many cases of these conjectures have been proved, in [Gee06b] and [GS] respectively, but the general case has so far been out of reach.

As far as we know, there is no corresponding conjecture in the literature for automorphic forms on quaternion algebras that are ramified at p (although the results in the case $F = \mathbb{Q}$ are easily deduced from the discussion in section 4 of [Kha01]). We specify a conjectural set of weights $W^?(\bar{\rho})$, depending only on the restrictions of $\bar{\rho}$ to decomposition subgroups at places dividing p . In the case that these restrictions are semisimple, the conjecture is completely explicit (and depends only on the restrictions to inertia subgroups). In the general case the set $W^?(\bar{\rho})$ is defined in terms of the existence of certain potentially Barsotti-Tate lifts of specific type, and so depends on some rather delicate questions involving extensions of crystalline characters. In fact, we always make a definition in terms of the existence of certain potentially Barsotti-Tate lifts, and in the semisimple case we make this description explicit by means of calculations with Breuil modules and strongly divisible modules.

We assume that p is odd, and that $\bar{\rho}|_{G_{F(\zeta_p)}}$ is irreducible. We make a mild additional assumption if $p = 5$. All of these restrictions are imposed by our use of the modularity lifting theorems of [Kis07b] (or rather, by their use in [Gee06a]). Under these assumptions, we are able to prove that if for each place $v|p$, $\bar{\rho}|_{G_{F_v}}$ is not of the form $\begin{pmatrix} \psi_1 & * \\ 0 & \psi_2 \end{pmatrix}$ with ψ_1/ψ_2 equal to the mod p cyclotomic character, then $W^?(\bar{\rho}) = W(\bar{\rho})$. In the exceptional cases we establish that $W^?(\bar{\rho}) \subset W(\bar{\rho})$, with equality up to a short list of possible exceptions (for example, in the case that there is only one place of F above p there is only one exception). Our techniques are analogous to those of [Gee06b] and [GS]. As in those papers, the plan is to construct modular lifts of $\bar{\rho}$ which are potentially semistable of specific type, using the techniques of Khare-Wintenberger, as explained in [Gee06a].

The significant advantage in the present situation over those considered in [GS] and [Gee06b] is that the property of being modular of a specific weight corresponds exactly to the property of having a lift of some specific type (in the case considered in [Gee06b] this correspondence was considerably weaker). Accordingly, we have no regularity assumption on the weights, we do not have to use Buzzard's "weight cycling" technique, and especially we do not need to make any restriction on the splitting behaviour of p in F .

In the case that the restrictions of $\bar{\rho}$ to decomposition groups at places dividing p are all semisimple, we establish an explicit description of $W^?(\bar{\rho})$ by a computation in two stages. In one step we make use of Breuil modules with descent data, in the same style as analogous computations in the literature; our calculations are rather more complicated than those made in the past, however, as we have no restrictions on the ramification or inertial degrees of our local fields.

For the second step, we have to exhibit potentially Barsotti-Tate lifts of the appropriate types. Writing down such lifts is rather non-trivial. We accomplish this by means of an explicit construction of corresponding strongly divisible modules; again, these calculations are more complicated than those in the literature, because we make no restrictions on the ramification or inertial degrees of our local fields.

We also note that while we work throughout with definite quaternion algebras, it should not be difficult to extend our results to indefinite algebras; one needs only to establish the analogue of Lemma 2.2 (see for example the proof of Lemma 6 of [DT94] for the case $F = \mathbb{Q}$).

We now detail the outline of the paper. In section 2 we give our initial definitions and notation. In particular, we introduce spaces of algebraic modular forms on definite quaternion algebras, and we explain what it means for $\bar{\rho}$ to be modular of a specific weight.

In section 3 we explain which tame lifts we will need to consider, and the relationship between the existence of modular lifts of specified types and the property of being modular of a certain weight. This amounts to recalling certain concrete instances of the local Langlands and Jacquet-Langlands correspondences for GL_2 and local-global compatibility. All of this material is completely standard.

Section 4 begins the work of establishing an explicit description of $W^?(\bar{\rho})$, by finding necessary conditions for the existence of potentially Barsotti-Tate lifts of particular type, via calculations with Breuil modules. The sufficiency of these conditions is established in section 6, by writing down explicit strongly divisible modules. Both sections make use of a lemma relating the type of the lifts to the descent data on the Breuil modules and strongly divisible modules, which is established in section 5.

These local calculations are summarised in section 7. Finally, in section 8 we prove our main theorem.

2. NOTATION AND ASSUMPTIONS

Let p be an odd prime. Fix an algebraic closure $\bar{\mathbb{Q}}$ of \mathbb{Q} , an algebraic closure $\bar{\mathbb{Q}}_p$ of \mathbb{Q}_p , and an embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. We will consider all finite extensions of \mathbb{Q} (respectively \mathbb{Q}_p) to be contained in $\bar{\mathbb{Q}}$ (respectively $\bar{\mathbb{Q}}_p$). If K is such an extension, we let G_K denote its absolute Galois group $\mathrm{Gal}(\bar{K}/K)$. Let F be a totally real field. Let $\bar{\rho} : G_F \rightarrow \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ be a continuous Galois representation. Assume from now on that $\bar{\rho}|_{G_{F(\zeta_p)}}$ is absolutely irreducible. If $p = 5$ and the projective image of $\bar{\rho}$ is isomorphic to $\mathrm{PGL}_2(\mathbb{F}_5)$, assume further that $[F(\zeta_5) : F] = 4$. We normalise the isomorphisms of local class field theory so that a uniformiser corresponds to a geometric Frobenius element.

We wish to discuss the Serre weights of $\bar{\rho}$ for quaternion algebras ramified at all places dividing p . We choose to work with totally definite quaternion algebras. We recall the basic definitions and results that we need.

Let D be a quaternion algebra with center F which is ramified at all infinite places of F and at a set Σ of finite places which contains all primes dividing p . Fix a maximal order \mathcal{O}_D of D and for each finite place $v \notin \Sigma$ fix an isomorphism $\mathcal{O}_{D,v} := (\mathcal{O}_D)_v \xrightarrow{\sim} M_2(\mathcal{O}_{F_v})$. For any finite place v let π_v denote a uniformiser of F_v .

Let $U = \prod_v U_v \subset (D \otimes_F \mathbb{A}_F^f)^\times$ be a compact open subgroup, with each $U_v \subset \mathcal{O}_{D,v}^\times$. Furthermore, assume that $U_v = \mathcal{O}_{D,v}^\times$ for all $v \in \Sigma$.

Take A a topological \mathbb{Z}_p -algebra. For each place $v|p$ fix a continuous representation $\sigma_v : U_v \rightarrow \mathrm{Aut}(W_v)$ with W_v a finite free A -module. Let σ denote the representation $\otimes_{v|p} \sigma_v$ of $U_p := \prod_{v|p} U_v$, acting on $W_\sigma := \otimes_{v|p} W_v$. We regard σ as a representation of U in the obvious way (that is, we let U_v act trivially if $v \nmid p$).

Fix also a character $\psi : F^\times \backslash (\mathbb{A}_F^f)^\times \rightarrow A^\times$ such that for any finite place v of F , $\sigma|_{U_v \cap \mathcal{O}_{F_v}^\times}$ is multiplication by ψ^{-1} . Then we can think of W_σ as a $U(\mathbb{A}_F^f)^\times$ -module by letting $(\mathbb{A}_F^f)^\times$ act via ψ^{-1} .

Let $S_{\sigma,\psi}(U, A)$ denote the set of continuous functions

$$f : D^\times \backslash (D \otimes_F \mathbb{A}_F^f)^\times \rightarrow W_\sigma$$

such that for all $g \in (D \otimes_F \mathbb{A}_F^f)^\times$ we have

$$f(gu) = \sigma(u)^{-1} f(g) \text{ for all } u \in U,$$

$$f(gz) = \psi(z) f(g) \text{ for all } z \in (\mathbb{A}_F^f)^\times.$$

We can write $(D \otimes_F \mathbb{A}_F^f)^\times = \prod_{i \in I} D^\times t_i U(\mathbb{A}_F^f)^\times$ for some finite index set I and some $t_i \in (D \otimes_F \mathbb{A}_F^f)^\times$. Then we have

$$S_{\sigma,\psi}(U, A) \xrightarrow{\sim} \bigoplus_{i \in I} W_\sigma^{(U(\mathbb{A}_F^f)^\times \cap t_i^{-1} D^\times t_i) / F^\times},$$

the isomorphism being given by the direct sum of the maps $f \mapsto f(t_i)$. From now on we make the following assumption:

$$\text{For all } t \in (D \otimes_F \mathbb{A}_F^f)^\times \text{ the group } (U(\mathbb{A}_F^f)^\times \cap t^{-1} D^\times t) / F^\times = 1.$$

One can always replace U by a subgroup (satisfying the above assumptions, and without changing U_p) for which this holds (*cf.* section 3.1.1 of [Kis07a]). Under this assumption $S_{\sigma,\psi}(U, A)$ is a finite free A -module, and the functor $W_\sigma \mapsto S_{\sigma,\psi}(U, A)$ is exact in W_σ .

We now define some Hecke algebras. Let S be a set of finite places containing Σ and the primes v of F such that $U_v \neq \mathcal{O}_{D,v}^\times$. Let $\mathbb{T}_{S,A}^{\text{univ}} = A[T_v, S_v]_{v \notin S}$ be the commutative polynomial ring in the formal variables T_v, S_v . Consider the left action of $(D \otimes_F \mathbb{A}_F^f)^\times$ on W_σ -valued functions on $(D \otimes_F \mathbb{A}_F^f)^\times$ given by $(gf)(z) = f(zg)$. Then we make $S_{\sigma,\psi}(U, A)$ a $\mathbb{T}_{S,A}^{\text{univ}}$ -module by letting S_v act via the double coset $U \begin{pmatrix} \pi_v & 0 \\ 0 & \pi_v \end{pmatrix} U$ and T_v via $U \begin{pmatrix} \pi_v & 0 \\ 0 & 1 \end{pmatrix} U$. These are independent of the choices of π_v . We will write $\mathbb{T}_{\sigma,\psi}(U, A)$ or $\mathbb{T}_{\sigma,\psi}(U)$ for the image of $\mathbb{T}_{S,A}^{\text{univ}}$ in $\text{End } S_{\sigma,\psi}(U, A)$.

Let \mathfrak{m} be a maximal ideal of $\mathbb{T}_{S,A}^{\text{univ}}$. We say that \mathfrak{m} is in the support of (σ, ψ) if $S_{\sigma,\psi}(U, A)_\mathfrak{m} \neq 0$. Now let \mathcal{O} be the ring of integers in $\overline{\mathbb{Q}}_p$, with residue field $\mathbb{F} = \overline{\mathbb{F}}_p$, and suppose that $A = \mathcal{O}$ in the above discussion, and that σ has open kernel. Consider a maximal ideal $\mathfrak{m} \subset \mathbb{T}_{S,\mathcal{O}}^{\text{univ}}$ with residue field \mathbb{F} which is induced by a maximal ideal of $\mathbb{T}_{\sigma,\psi}(U, \mathcal{O})$. Then there is a semisimple Galois representation $\bar{\rho}_\mathfrak{m} : G_F \rightarrow \text{GL}_2(\mathbb{F})$ associated to \mathfrak{m} which is characterised up to conjugacy by the property that if $v \notin S$ then $\bar{\rho}_\mathfrak{m}|_{G_{F_v}}$ is unramified, and if Frob_v is an arithmetic Frobenius at v then the trace of $\bar{\rho}_\mathfrak{m}(\text{Frob}_v)$ is the image of T_v in \mathbb{F} .

We are now in a position to define what it means for a representation to be modular of some weight. Let $v|p$ be a place of F , so that $U_v = \mathcal{O}_{D,v}^\times$. Let σ_v be an irreducible \mathbb{F} -representation of U_v . Note that if Π_v is a uniformiser of $\mathcal{O}_{D,v}$, then $k_{2,v} := \mathcal{O}_{D,v}/\Pi_v$ is a finite field, a quadratic extension of the residue field k_v of F_v . The kernel of the reduction map $U_v \rightarrow k_{2,v}^\times$ is a pro- p group, so σ_v factors through this kernel, and is a representation of the finite abelian group $k_{2,v}^\times$. It is therefore one-dimensional. Let $\sigma = \otimes_{v|p} \sigma_v$.

Definition 2.1. We say that $\bar{\rho}$ is modular of weight σ if for some D, S, U, ψ , and \mathfrak{m} as above, we have $S_{\sigma,\psi}(U, \mathbb{F})_\mathfrak{m} \neq 0$ and $\bar{\rho}_\mathfrak{m} \cong \bar{\rho}$.

Assume from now on that $\bar{\rho}$ is modular of some weight, and fix $D, S, U, \psi, \mathfrak{m}$ as in the definition. Write $W(\bar{\rho})$ for the set of weights σ for which $\bar{\rho}$ is modular of weight σ .

One can gain information about the weights associated to a particular Galois representation by considering lifts to characteristic zero. The key is the following basic lemma.

Lemma 2.2. *Let $\psi : F^\times \backslash (\mathbb{A}_F)^\times \rightarrow \mathcal{O}^\times$ be a continuous character, and write $\bar{\psi}$ for the composite of ψ with the projection $\mathcal{O}^\times \rightarrow \mathbb{F}^\times$. Fix a representation σ' of U_p on a finite free \mathcal{O} -module $W_{\sigma'}$, and an irreducible representation σ of U_p on a finite free \mathbb{F} -module W_σ . Suppose that we have $\sigma'|_{U_v \cap \mathcal{O}_{F_v}^\times} = \psi^{-1}|_{U_v \cap \mathcal{O}_{F_v}^\times}$ and $\sigma|_{U_v \cap \mathcal{O}_{F_v}^\times} = \bar{\psi}^{-1}|_{U_v \cap \mathcal{O}_{F_v}^\times}$ for all finite places v .*

Let \mathfrak{m} be a maximal ideal of $\mathbb{T}_{S, \mathcal{O}}^{\text{univ}}$.

Suppose that W_σ occurs as a U_p -module subquotient of $W_{\sigma'} \otimes \mathbb{F}$. If \mathfrak{m} is in the support of $(\sigma, \bar{\psi})$, then \mathfrak{m} is in the support of (σ', ψ) .

Conversely, if \mathfrak{m} is in the support of (σ', ψ) , then \mathfrak{m} is in the support of $(\sigma, \bar{\psi})$ for some irreducible U_p -module subquotient $W_{\sigma'}$ of $W_{\sigma'} \otimes \mathbb{F}$.

Proof. The first part is proved just as in Lemma 3.1.4 of [Kis07b], and the second part follows from Proposition 1.2.3 of [AS86]. \square

We also have the following useful lemma, which was first observed by Serre in the case $F = \mathbb{Q}$ (see remark (11) in Serre's letter to Tate in [Ser96]). For each place $v|p$ we let q_v denote the order of the residue field k_v of F_v . If σ_v is an irreducible \mathbb{F} -representation of $\mathcal{O}_{D, v}^\times$, then by the remarks above it is a character of $k_{2, v}^\times$. Thus $\sigma_v^{q_v}$ is another irreducible \mathbb{F} -representation, and $\sigma_v^{q_v^2} = \sigma_v$. Serre observed that the set $W(\bar{\rho})$ is preserved by this operation. This is essentially a consequence of the structure of $\mathcal{O}_{D, v}$. Let $K_{2, v} = W(k_{2, v})[1/p]$, a subfield of D_v . Note that there is a choice of uniformiser Π_v of D_v with the property that conjugation by Π_v preserves $K_{2, v}$, and acts on it via a non-trivial involution. In particular, the induced action on $k_{2, v}$ is via the q_v -th power map.

Lemma 2.3. *Let v be a place of F dividing p , and let σ be a weight as above. Let $\sigma' = \sigma_v^{q_v} \otimes_{w|p, w \neq v} \sigma_w$. Then $\bar{\rho}$ is modular of weight σ if and only if it is modular of weight σ' .*

Proof. It suffices to exhibit a bijection

$$\theta : S_{\sigma, \psi}(U, \mathbb{F}) \rightarrow S_{\sigma', \psi}(U, \mathbb{F})$$

which commutes with the action of $\mathbb{T}_{S, \mathcal{O}}^{\text{univ}}$. Let $\Pi \in (D \otimes_F \mathbb{A}^f)^\times$ be trivial away from v , and equal to Π_v at v , where Π_v is as in the previous paragraph. Then we define θ by

$$(\theta f)(x) := f(x\Pi).$$

It is straightforward to check that this map has the required properties; the key point is that if $u \in U$, then

$$\begin{aligned}
(\theta f)(xu) &= f(xu\Pi) \\
&= f(x\Pi(\Pi^{-1}u\Pi)) \\
&= \sigma(\Pi^{-1}u\Pi)^{-1}f(x\Pi) \\
&= \sigma'(u)^{-1}f(x\Pi) \\
&= \sigma'(u)^{-1}(\theta f)(x).
\end{aligned}$$

□

3. WEIGHTS ARE CONTROLLED BY LIFTS OF TAME TYPE

We distinguish two types of irreducible \mathbb{F} -representations σ_v of U_v . Recall that any such representation is 1-dimensional, and factors through $k_{2,v}^\times$, with $k_{2,v}$ a quadratic extension of k_v .

Definition 3.1. We say that σ_v is of type I if it does not factor through the norm $k_{2,v}^\times \rightarrow k_v^\times$. Otherwise, we say that it is of type II.

We now recall some facts about the local Langlands and local Jacquet-Langlands correspondences. Let K be a finite extension of \mathbb{Q}_p , let L be an unramified quadratic extension of K , and let D be a nonsplit quaternion algebra over K . Consider L as a subfield of D . Let k be the residue field of K , of cardinality q . If π is an irreducible admissible representation of D^\times , we let $JL(\pi)$ be the corresponding representation of $\mathrm{GL}_2(K)$. If π is an irreducible admissible representation of $\mathrm{GL}_2(K)$, we let $LL(\pi)$ denote the corresponding representation of the Weil group W_K of K . Let $N_D : D^\times \rightarrow K^\times$ be the reduced norm. As usual we identify characters of L^\times or K^\times with characters of the corresponding Weil groups via local class field theory. If χ is a character of L^\times which does not factor through the norm to K^\times , we denote the corresponding supercuspidal representation of $\mathrm{GL}_2(K)$ by W_χ .

- If χ is a character of K^\times , then $JL(\chi \circ N_D) = (\chi \circ \det) \otimes St$, where St is the Steinberg representation.
- Suppose that χ is a character of L^\times of conductor 1. Then $LL(\pi)|_{I_K} \cong \chi|_{I_K} \oplus \chi|_{I_K}^q$ if and only if $\pi = W_{\chi'}$ for some unramified twist χ' of χ . [See section A.3.2 of [BM02].]
- If χ is a character of L^\times of conductor 1, then $JL^{-1}(W_\chi)$ is 2-dimensional. Furthermore, $JL^{-1}(W_\chi)|_{\mathcal{O}_L^\times} \cong \chi|_{\mathcal{O}_L^\times} \oplus \chi|_{\mathcal{O}_L^\times}^q$. [See section 7 of [Pra90].]

We now recall some definitions relating to potentially semistable lifts of particular type. We use the conventions of [Sav05].

Definition 3.2. Let τ_v be an inertial type. We say that a lift ρ of $\bar{\rho}|_{G_{F_v}}$ is parallel potentially Barsotti-Tate (respectively parallel potentially semistable) of type τ_v if ρ is potentially Barsotti-Tate (respectively potentially semistable with all Hodge-Tate weights equal to 0 or 1), has determinant a finite order character of order prime to p times the cyclotomic character, and the corresponding Weil-Deligne representation, when restricted to I_{F_v} , is isomorphic to τ_v .

Note that for a two-dimensional de Rham representation with all Hodge-Tate weights equal to 0 or 1, the condition that all pairs of labeled Hodge-Tate weights

are $\{0, 1\}$ is equivalent to the condition that the determinant is the product of the cyclotomic character, a finite order character, and an unramified character; the condition of being parallel is slightly stronger than this.

Definition 3.3. Let $v|p$. We say that a representation $\rho : G_{F_v} \rightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ is ordinary if $\rho|_{I_{F_v}}$ is an extension of a finite order character by a finite order character times the cyclotomic character.

We let τ_v denote the inertial type of I_{F_v} given by $\tilde{\sigma}_v \oplus \tilde{\sigma}_v^{q_v}$, where a tilde denotes a Teichmüller lift (considered as a representation of I_{F_v} via local class field theory).

Lemma 3.4. $\bar{\rho}$ is modular of weight $\sigma = \otimes_{v|p} \sigma_v$ if and only if $\bar{\rho}$ lifts to a modular Galois representation which for all places $v|p$ is parallel potentially Barsotti-Tate of type τ_v at v if σ_v is of type I, and is parallel potentially semistable of type τ_v at v , but not potentially crystalline, if σ_v is of type II.

Proof. We first tackle the only if direction. We apply Lemma 2.2, with \mathfrak{m} a maximal ideal with $\bar{\rho}_{\mathfrak{m}} \cong \bar{\rho}$ and $S_{\sigma, \psi}(U, \mathbb{F})_{\mathfrak{m}} \neq 0$ (such a maximal ideal exists by the assumption that $\bar{\rho}$ is modular of weight σ), and

$$\sigma' = \otimes_{v|p} \sigma'_v$$

where

$$\sigma'_v = (JL^{-1}(W_{\tilde{\sigma}_v}))|_{\mathcal{O}_D^\times}$$

if σ_v is of type I, and

$$\sigma'_v = \tilde{\sigma}_v$$

if σ_v is of type II. The correspondence between the algebraic modular forms considered in section 2 and automorphic representations of D^\times is explained in section 3.1.14 of [Kis07b] (there is a running assumption in that paper that D is split at all places dividing p , but it is not needed in this discussion, and if one sets the representation $W_{\tau^{\mathrm{alg}}}$ of *loc. cit.* to be the trivial representation the discussion goes through immediately in our case), and we see that there is an automorphic representation π of D^\times whose weight is the trivial representation, whose Hecke polynomials at unramified places lift the characteristic polynomials of the corresponding Frobenius elements for $\bar{\rho}$, and such that for each place $v|p$, π_v is either $JL^{-1}(W_\chi)$ for χ an unramified twist of $\tilde{\sigma}_v$ if σ_v is of type I, or an unramified twist of $\tilde{\sigma}_v$ if σ_v is of type II.

Applying the Jacquet-Langlands correspondence (Theorem 16.1 of [JL70]) we see that there is an automorphic representation π' of $\mathrm{GL}_2(\mathbb{A}_F)$ with the same infinitesimal character as the trivial representation, whose Hecke polynomials at unramified places lift the characteristic polynomials of the corresponding Frobenius elements for $\bar{\rho}$, and such that for each place $v|p$, π_v is either W_χ for χ an unramified twist of $\tilde{\sigma}_v$ if σ_v is of type I, or an unramified twist of $(\tilde{\sigma}_v \circ \det) \otimes St$ if σ_v is of type II. The compatibility of the local and global Langlands correspondences at places dividing p (see [Kis08]), and the results on the form of the local Langlands correspondence recalled above, show that the Galois representation corresponding to π' gives a representation of the required form.

For the converse, we may reverse the above argument, and we see that Lemma 2.2 guarantees that $\bar{\rho}$ is modular of a weight $\mu = \otimes_{v|p} \mu_v$, where for each $v|p$ if σ_v is of type II then $\mu_v = \sigma_v$, and if σ_v is of type I then $\mu_v = \sigma_v$ or $\sigma_v^{q_v}$. The result then follows from Lemma 2.3. \square

This motivates the following definition of $W^?(\bar{\rho})$.

Definition 3.5. For each place $v|p$, let $W^?(\bar{\rho}|_{G_{F_v}})$ denote the set of σ_v of type I such that $\bar{\rho}|_{G_{F_v}}$ has a parallel potentially Barsotti-Tate lift of type τ_v , together with the set of σ_v of type II such that $\bar{\rho}|_{G_{F_v}}$ has a parallel potentially semistable lift of type τ_v which is not potentially crystalline. Let $W^?(\bar{\rho})$ be the set of weights $\sigma = \otimes_{v|p} \sigma_v$ with $\sigma_v \in W^?(\bar{\rho}|_{G_{F_v}})$ for all $v|p$.

Note that by Lemma 3.4 we have $W(\bar{\rho}) \subset W^?(\bar{\rho})$. We will prove under a mild hypothesis that $W(\bar{\rho}) = W^?(\bar{\rho})$ in section 8. In the intervening sections we will give an explicit description of $W^?(\bar{\rho})$ in the case that $\bar{\rho}|_{G_{F_v}}$ is semisimple for each $v|p$. It is already possible to see that weights of type II are rather rare.

Lemma 3.6. *If $\bar{\rho}$ is modular of weight $\sigma = \otimes_{v|p} \sigma_v$, and σ_v is of type II, then*

$$\bar{\rho}|_{I_{F_v}} \cong \sigma_v \begin{pmatrix} \epsilon & * \\ 0 & 1 \end{pmatrix}$$

where σ_v is regarded as a character of I_{F_v} via local class field theory, and ϵ is the cyclotomic character.

Proof. This follows from Lemma 3.4, and the well-known fact that 2-dimensional semistable non-crystalline p -adic representations with all pairs of labeled Hodge-Tate weights equal to $\{0, 1\}$ are unramified twists of an extension of the trivial character by the cyclotomic character. \square

4. NECESSARY CONDITIONS

4.1. Breuil modules with descent data. Let k be a finite extension of \mathbb{F}_p , define $K_0 = W(k)[1/p]$, and let K be a finite totally ramified extension of K_0 of degree e' . Suppose that L is a p -adic subfield of K such that K/L is Galois and tamely ramified. Assume further that there is a uniformiser π of \mathcal{O}_K such that $\pi^{e(K/L)} \in L$, where $e(K/L)$ is the ramification degree of K/L , and fix such a π . Since K/L is tamely ramified, the category of Breuil modules with coefficients and descent data is easy to describe (see [Sav08]). Let k_E be a finite extension of \mathbb{F}_p . The category $\text{BrMod}_{\text{dd}, L}$ consists of quadruples $(\mathcal{M}, \text{Fil}^1 \mathcal{M}, \phi_1, \{\widehat{g}\})$ where:

- \mathcal{M} is a finitely generated $(k \otimes_{\mathbb{F}_p} k_E)[u]/u^{e'p}$ -module, free over $k[u]/u^{e'p}$.
- $\text{Fil}^1 \mathcal{M}$ is a $(k \otimes_{\mathbb{F}_p} k_E)[u]/u^{e'p}$ -submodule of \mathcal{M} containing $u^{e'} \mathcal{M}$.
- $\phi_1 : \text{Fil}^1 \mathcal{M} \rightarrow \mathcal{M}$ is k_E -linear and ϕ -semilinear (where $\phi : k[u]/u^{e'p} \rightarrow k[u]/u^{e'p}$ is the p -th power map) with image generating \mathcal{M} as a $(k \otimes_{\mathbb{F}_p} k_E)[u]/u^{e'p}$ -module.
- $\widehat{g} : \mathcal{M} \rightarrow \mathcal{M}$ are additive bijections for each $g \in \text{Gal}(K/L)$, preserving $\text{Fil}^1 \mathcal{M}$, commuting with the ϕ -, and k_E -actions, and satisfying $\widehat{g}_1 \circ \widehat{g}_2 = \widehat{g_1 \circ g_2}$ for all $g_1, g_2 \in \text{Gal}(K/L)$, and $\widehat{1}$ is the identity. Furthermore, if $a \in k \otimes_{\mathbb{F}_p} k_E$, $m \in \mathcal{M}$ then $g(au^i m) = g(a)((g(\pi)/\pi)^i \otimes 1)u^i g(m)$.

The category $\text{BrMod}_{\text{dd}, L}$ is equivalent to the category of finite flat group schemes over \mathcal{O}_K together with a k_E -action and descent data on the generic fibre from K to L (this equivalence depends on π).

We choose in this paper to adopt the conventions of [BM02] and [Sav05], rather than those of [BCDT01]; thus rather than working with the usual contravariant

equivalence of categories, we work with a covariant version of it, so that our formulae for generic fibres will differ by duality and a twist from those following the conventions of [BCDT01]. To be precise, we obtain the associated G_L -representation (which we will refer to as the generic fibre) of an object of $\mathrm{BrMod}_{\mathrm{dd},L}$ via the functor $T_{\mathrm{st},2}^L$.

Let E be a finite extension of \mathbb{Q}_p with integers \mathcal{O}_E , maximal ideal \mathfrak{m}_E , and residue field k_E . Recall from [Sav05, Sec. 2] that the functor $D_{\mathrm{st},2}^K$ is an equivalence of categories between the category of E -representations of G_L which are semistable when restricted to G_K and have Hodge-Tate weights in $\{0, 1\}$, and the category of weakly admissible filtered (ϕ, N) -modules D with descent data and E -coefficients such that $\mathrm{Fil}^0(K \otimes_{K_0} D) = K \otimes_{K_0} D$ and $\mathrm{Fil}^2(K \otimes_{K_0} D) = 0$.

Suppose that ρ is a representation in the source of $D_{\mathrm{st},2}^K$. Write $S = S_{K, \mathcal{O}_E}$ (notation and terminology in this paragraph are as in [Sav05, Sec. 4]). Then $T_{\mathrm{st},2}^L$ is an equivalence of categories between strongly divisible modules \mathcal{M} (with \mathcal{O}_E -coefficients and descent data) in $S[1/p] \otimes_{K_0 \otimes E} D_{\mathrm{st},2}^K(\rho)$ and Galois-stable \mathcal{O}_E -lattices in ρ . This equivalence is compatible with reduction mod \mathfrak{m}_E , so that applying $T_{\mathrm{st},2}^L$ to the object $(k \otimes_{\mathbb{F}_p} k_E)[u]/(u^{e^p}) \otimes_{S/\mathfrak{m}_E S} (\mathcal{M}/\mathfrak{m}_E \mathcal{M})$ of $\mathrm{BrMod}_{\mathrm{dd},L}$ yields a reduction mod p of ρ (see [Sav05, Cor. 4.12, Prop 4.13]).

To simplify notation, for the remainder of the paper we write simply $\mathcal{M}/\mathfrak{m}_E \mathcal{M}$ for the above reduction mod \mathfrak{m}_E of \mathcal{M} in $\mathrm{BrMod}_{\mathrm{dd},L}$ (we will never mean the literal $S/\mathfrak{m}_E S$ -module). Let ℓ be the residue field of L , and let ur_λ denote the unramified character of G_L sending an arithmetic Frobenius element to λ . Define $N_{\ell/\mathbb{F}_p, k_E} : \ell \otimes_{\mathbb{F}_p} k_E \rightarrow \mathbb{F}_p \otimes_{\mathbb{F}_p} k_E \cong k_E$ to be the norm map $x \mapsto \prod_{\beta \in \mathrm{Gal}(\ell/\mathbb{F}_p)} \beta(x)$, with each β acting trivially on k_E .

The following lemma is a more precise version of [GS, Lem. 5.2].

Lemma 4.1. *Let $\bar{\chi} : \mathrm{Gal}(K/L) \rightarrow k_E^\times$ be a character, and for $c \in (\ell \otimes_{\mathbb{F}_p} k_E)^\times$ let $\mathcal{M}(\bar{\chi}, c)$ denote the free of rank one Breuil module with k_E -coefficients and descent data from K to L with generator v and*

$$\mathrm{Fil}^1 \mathcal{M}(\bar{\chi}, c) = \mathcal{M}(\bar{\chi}, c), \quad \phi_1(v) = cv, \quad \hat{g}(v) = (1 \otimes \bar{\chi}(g))v$$

for $g \in \mathrm{Gal}(K/L)$. Then $T_{\mathrm{st},2}^L(\mathcal{M}(\bar{\chi}, c)) = \mathrm{ur}_\lambda \cdot \bar{\chi}$, where $\lambda = N_{\ell/\mathbb{F}_p, k_E}(c)^{-1}$.

Proof. This statement is exactly the same as [GS, Lem. 5.2], except that here we determine the unramified character multiplying $\bar{\chi}$. We return ourselves to the proof of that statement, and in particular we re-adopt the notation from that proof, so that χ is the Teichmüller lift of $\bar{\chi}$, the element $\tilde{c} \in (W(\ell) \otimes_{\mathbb{Z}_p} \mathcal{O}_E)^\times$ is a lift of c , and $D := D(\chi, \tilde{c})$ is a filtered ϕ -module of rank one over $K_0 \otimes E$ with descent data χ and generator \mathbf{v} such that $\phi(\mathbf{v}) = p\tilde{c}\mathbf{v}$.

The representation $V_{\mathrm{st},2}^L(D)$ giving rise to D is contained in $(B_{\mathrm{st}} \otimes_{K_0} D)^{\phi=p}$ (see the definition after [Sav05, Cor. 2.10]), and is generated by some $\alpha\mathbf{v}$ with $\alpha \in B_{\mathrm{st}} \otimes_{K_0} E$; then $p\alpha\mathbf{v} = \phi(\alpha\mathbf{v}) = \phi(\alpha)\tilde{c}p\mathbf{v}$, so that $\phi(\alpha)\tilde{c} = \alpha$. If $f = [\ell : \mathbb{F}_p]$ it follows that $\phi^{(f)}(\alpha) = N_{L_0/\mathbb{Q}_p, E}(\tilde{c})^{-1}\alpha$, where $L_0 = W(\ell)[1/p]$ and $N_{L_0/\mathbb{Q}_p, E}$ is defined via the obvious analogy with $N_{\ell/\mathbb{F}_p, k_E}$. Set $\tilde{\lambda} = N_{L_0/\mathbb{Q}_p, E}(\tilde{c})^{-1}$.

Now if $g \in G_L$ is a lift of an n th power of arithmetic Frobenius, with $n \in \mathbb{Z}$, then $g(\alpha\mathbf{v}) = \phi^{(nf)}(\alpha)\chi(g)\mathbf{v} = (\mathrm{ur}_{\tilde{\lambda}} \cdot \chi)(g)\alpha\mathbf{v}$. Since $\tilde{\lambda}$ lifts λ , the result follows by continuity. \square

For the remainder of this paper we make the hypothesis that k_E is sufficiently large as to contain an embedding of k . Let σ_0 be a fixed choice of embedding

$k \hookrightarrow k_E$ and recursively define $\sigma_{i+1}^p = \sigma_i$. If M is any $(k \otimes_{\mathbb{F}_p} k_E)$ -module, we recall from [Sav08] that M decomposes as a direct sum $M = \bigoplus_{i=0}^{d-1} M_i$, where M_i is the k_E -submodule on which multiplication by $x \otimes 1$ for $x \in k$ is the same as multiplication by $1 \otimes \sigma_i(x)$. In fact there is a collection of idempotents $e_i \in k \otimes_{\mathbb{F}_p} k_E$ so that $M_i = e_i M$ and $\phi(e_i) = e_{i+1}$.

Suppose now that \mathcal{M} is an object of $\text{BrMod}_{\text{dd},L}$. Note that ϕ_1 maps $(\text{Fil}^1 \mathcal{M})_i$ into \mathcal{M}_{i+1} . For $g \in G_L$ let $\bar{\eta}(g)$ be the image of $g(\pi)/\pi$ in (the $e(K/L)$ th roots of unity of) k . The rank one objects of $\text{BrMod}_{\text{dd},L}$ are classified as follows.

Proposition 4.2 ([Sav08], Theorem 3.5). *With our fixed choice of uniformiser π , every rank one object of $\text{BrMod}_{\text{dd},L}$ with descent data relative to L has the form:*

- $\mathcal{M} = ((k \otimes_{\mathbb{F}_p} k_E)[u]/u^{e'}) \cdot v$,
- $(\text{Fil}^1 \mathcal{M})_i = u^{r_i} \mathcal{M}_i$,
- $\phi_1(\sum_{i=0}^{d-1} u^{r_i} e_i v) = cv$ for some $c \in (\ell \otimes_{\mathbb{F}_p} k_E)^\times$, and
- $\hat{g}(v) = \sum_{i=0}^{d-1} (\bar{\eta}(g)^{k_i} \otimes 1) e_i v$ for all $g \in \text{Gal}(K/L)$,

where $0 \leq r_i \leq e'$ and $0 \leq k_i < e(K/L)$ are sequences of integers satisfying $k_i \equiv p(k_{i-1} + r_{i-1}) \pmod{e(K/L)}$; furthermore the sequences r_i, k_i are periodic with period dividing $f = [\ell : \mathbb{F}_p]$.

Corollary 4.3. *In the above proposition, suppose that $e(K/L)$ is divisible by $p^f - 1$. Define $s_0 = p(p^{f-1}r_0 + \dots + r_{f-1})/(p^f - 1)$ and $\lambda = \text{N}_{\ell/\mathbb{F}_p, k_E}(c)^{-1}$. Then $T_{\text{st},2}^L(\mathcal{M}) = (\sigma_0 \circ \bar{\eta}^{k_0+s_0}) \cdot \text{ur}_\lambda$.*

Remark 4.4. According to [Sav08, Rem. 3.6], the congruences $k_i \equiv p(k_{i-1} + r_{i-1}) \pmod{e(K/L)}$ imply that

$$p^{f-1}r_0 + \dots + r_{f-1} \equiv 0 \pmod{p^f - 1},$$

and k_0 is a solution to $-p(p^{f-1}r_0 + \dots + r_{f-1}) \equiv (p^f - 1)k_0 \pmod{e(K/L)}$. It follows that s_0 is an integer; moreover $(p^f - 1)(k_0 + s_0) \equiv 0 \pmod{e(K/L)}$, so that the image of $\bar{\eta}^{k_0+s_0}$ lies in ℓ^\times and $\bar{\eta}^{k_0+s_0}$ is actually a character.

Proof. We proceed as in Example 3.7 of [Sav08].

Define $s_i = p(r_i p^{f-1} + \dots + r_{i+f-1})/(p^f - 1)$ with subscripts taken modulo f , and observe that $(k_i + s_i) \equiv p^i(k_0 + s_0) \pmod{e(K/L)}$. Let $\bar{\chi} = \sigma_0 \circ \bar{\eta}^{k_0+s_0}$. We check that there is a morphism $\mathcal{M}(\bar{\chi}, c) \rightarrow \mathcal{M}$ with $\mathcal{M}(\bar{\chi}, c)$ as in Lemma 4.1 (except that here we will use w to denote its generator, since v is now our generator of \mathcal{M}).

The morphism will send $w \mapsto \sum_i u^{s_i} e_i v$. One checks easily that this is a morphism of Breuil modules. Indeed: the filtration is preserved since $s_i \geq r_i$; the morphism commutes with ϕ_1 because $s_{i+1} = p(s_i - r_i)$; and to check that the morphism commutes with descent data, use the fact that $\hat{g}(w) = (1 \otimes (\sigma_0 \circ \bar{\eta}^{k_0+s_0}(g)))w = \sum_i (\bar{\eta}^{k_i+s_i}(g) \otimes 1) e_i w$.

Now the claim follows immediately from Lemma 4.1 and an application of [Sav04, Prop. 8.3]. \square

4.2. Necessary conditions: the reducible case. Let \mathfrak{p} be a prime of F lying above p , and $\pi_{\mathfrak{p}} \in \mathfrak{p}$ our chosen uniformiser. Suppose that the residue field of $F_{\mathfrak{p}}$ has order $q = p^f$.

In this subsection and the next, we consider the following situation. Let L be the unramified quadratic extension of $F_{\mathfrak{p}}$, and K the splitting field of $u^{p^{2f}-1} - \pi_{\mathfrak{p}}$ over L . Let ϖ be a choice of $\pi_{\mathfrak{p}}^{1/(p^{2f}-1)}$ in K . Let k denote the residue field of

K , and if $g \in \text{Gal}(K/F_{\mathfrak{p}})$ then as before we let $\bar{\eta}(g)$ be the image of $g(\varpi)/\varpi$ in k . Suppose that $F_{\mathfrak{p}}$ has absolute ramification index e , and write $e' = e(p^{2f} - 1)$.

Suppose that $\bar{\rho} : G_{F_{\mathfrak{p}}} \rightarrow \text{GL}_2(k_E)$ with k_E a finite field into which k may be embedded, and assume that $\bar{\rho}$ is the reduction mod \mathfrak{m}_E of a parallel potentially Barsotti-Tate representation ρ of type $\chi \oplus \chi^q$, where $\chi \neq \chi^q$. Let \mathcal{H} be the \mathfrak{m}_E -torsion of the Barsotti-Tate group over \mathcal{O}_K corresponding to ρ ; then \mathcal{H} is a finite flat group scheme over \mathcal{O}_K with descent data to $F_{\mathfrak{p}}$ of the form $\bar{\chi} \oplus \bar{\chi}^q$, and $\bar{\rho}$ is the generic fibre of \mathcal{H} .

In this subsection we suppose that $\bar{\rho} \cong \begin{pmatrix} \psi_1 & * \\ 0 & \psi_2 \end{pmatrix}$, and we wish to determine the possibilities for ψ_1 and ψ_2 . We do this by means of a calculation with the corresponding Breuil modules. Note that by a standard scheme-theoretic closure argument, ψ_1 corresponds to a subgroup scheme \mathcal{G} of \mathcal{H} .

By Proposition 4.2, the Breuil module \mathcal{M} with descent data to $F_{\mathfrak{p}}$ corresponding to \mathcal{G} may be written in the form

- $\mathcal{M} = ((k \otimes_{\mathbb{F}_p} k_E)[u]/u^{e(p^{2f}-1)p}) \cdot v$,
- $(\text{Fil}^1 \mathcal{M})_i = u^{r_i} \mathcal{M}_i$,
- $\phi_1(u^{r_i} e_i v) = (1 \otimes \gamma_i) e_{i+1} v$ for some $\gamma_i \in k_E^\times$,
- $\hat{g}(\sum_{i=0}^{2f-1} e_i v) = \sum_{i=0}^{2f-1} (\bar{\eta}(g)^{k_i} \otimes 1) e_i v$ for all $g \in \text{Gal}(K/F_{\mathfrak{p}})$.

Here the k_i, e_i are integers with $k_i \in [0, p^{2f} - 1]$ and $r_i \in [0, e(p^{2f} - 1)]$ satisfying $k_{i+1} \equiv p(k_i + r_i) \pmod{p^{2f} - 1}$, and $r_{i+f} = r_i, k_{i+f} = k_i$; also $\gamma_{i+f} = \gamma_i$. We may also write $\hat{g}(e_i v) = (1 \otimes \chi_i(g)) e_i v$ where $\chi_i = \sigma_i \circ \bar{\eta}^{k_i}$, so that $\chi_i^q = \chi_{i+f}$.

The possibilities for k_i are prescribed by the fact that the descent data for \mathcal{G} is of the form $\bar{\chi} \oplus \bar{\chi}^q$, so that we have $\chi_i \in \{\bar{\chi}, \bar{\chi}^q\}$ for all i . (This follows for instance by Corollary 5.2, which does not depend on anything in this paper before Section 5. Note that this is the only place in this subsection where we make use of the hypothesis that $\bar{\rho}$ with the given descent data lifts to ρ , and this is only for convenience.) Let $\eta_i = (\sigma_i \circ \bar{\eta})|_{I_{F_{\mathfrak{p}}}}$ for $0 \leq i < 2f$ be a system of fundamental characters of niveau $2f$ of $I_{F_{\mathfrak{p}}}$; then $\omega_i = \eta_i \eta_{i+f}$ is a system of fundamental characters of niveau f . Write

$$\bar{\chi} = \prod_{i=0}^{2f-1} \eta_i^{c_i}$$

with $0 \leq c_i \leq p-1$. Then we see that each k_i is congruent modulo $p^{2f} - 1$ to either

$$c_i + p^{2f-1} c_{i+1} + p^{2f-2} c_{i+2} + \cdots + p c_{i-1}$$

or

$$c_{i+f} + p^{2f-1} c_{i+f+1} + p^{2f-2} c_{i+f+2} + \cdots + p c_{i+f-1}$$

but (since $\bar{\chi} \neq \bar{\chi}^q$) not both. We let J be the set of places such that the first possibility obtains, or equivalently, such that $\chi_i = \bar{\chi}$. The conditions $k_i = k_{i+f}$ and $\bar{\chi} \neq \bar{\chi}^q$ together imply that $i \in J$ if and only if $i+f \notin J$. We now consider the congruence $k_{i+1} \equiv p(k_i + r_i) \pmod{p^{2f} - 1}$. If $i \in J$ and $i+1 \in J$, or if $i \notin J$ and $i+1 \notin J$, this gives $r_i \equiv 0 \pmod{p^{2f} - 1}$. In either case, write $r_i = (p^{2f} - 1)x_i$, for some $0 \leq x_i \leq e$. If $i \in J$ and $i+1 \notin J$, we see that

$$r_i = (p^{2f} - 1)x_i + (p^f - 1)(p^{f-1}(c_{i+f+1} - c_{i+1}) + p^{f-2}(c_{i+f+2} - c_{i+2}) + \cdots + (c_i - c_{i+f}))$$

for some x_i , and if $i \notin J$ and $i+1 \in J$, then

$$r_i = (p^{2f} - 1)x_i + (p^f - 1)(p^{f-1}(c_{i+1} - c_{i+f+1}) + p^{f-2}(c_{i+2} - c_{i+f+2}) + \cdots + (c_{i+f} - c_i)).$$

In these cases we will determine the ranges for x_i shortly, but in any case note that $x_i = x_{i+f}$.

Following Corollary 4.3 we define

$$s_0 = \frac{p}{(p^{2f} - 1)} (r_0 p^{2f-1} + r_1 p^{2f-2} + \cdots + r_{2f-1}),$$

so that $\psi_1|_{I_{F_p}} = \bar{\eta}_0^{k_0 + s_0}$. Consider s_0 as a linear expression in the c_i . We begin by determining the coefficient of c_0 . It is elementary to check that this coefficient is p^f if $0 \in J$, and $-p^f$ otherwise. Indeed, for each transition $i \in J$, $i+1 \notin J$ with $i \in [0, f)$, the terms r_i and r_{i+f} together contribute p^f to the coefficient; on the other hand each transition $i \notin J$, $i+1 \in J$ in that range contributes $-p^f$. Since the net number transitions out of J from $i = 0$ to $i = f$ is 1 or -1 depending on whether $0 \in J$ or $0 \notin J$, the claim follows.

Thus the coefficient of c_0 in $k_0 + s_0$ is $p^f + 1$ if $0 \in J$, and 0 otherwise. By cyclic symmetry, the coefficient of c_i in $k_0 + s_0$ is $p^{2f-i}(p^f + 1)$ if $i \in J$, and 0 if $i \notin J$. We thus see that

$$\psi_1|_{I_{F_p}} = \prod_{i \in J} \omega_{\pi(i)}^{c_i} \prod_{i=0}^{f-1} \omega_i^{x_i}$$

where π is the natural projection from $\mathbb{Z}/2f\mathbb{Z}$ to $\mathbb{Z}/f\mathbb{Z}$.

It remains to determine the possible values of the x_i . As already remarked, if $i \in J$ and $i+1 \in J$, or if $i \notin J$ and $i+1 \notin J$, then we may take any $0 \leq x_i \leq e$. Since the expression $(p^f - 1)(p^{f-1}(c_{i+f+1} - c_{i+1}) + p^{f-2}(c_{i+f+2} - c_{i+2}) + \cdots + (c_i - c_{i+f}))$ is nonzero and is strictly between $1 - p^{2f}$ and $p^{2f} - 1$, in the other cases we allow either $0 \leq x_i \leq e - 1$ or $1 \leq x_i \leq e$, depending on whether the sign of this expression is positive or negative. If $i \in J$ and $i+1 \notin J$, then the allowable range is $0 \leq x_i \leq e - 1$ precisely when there is a $j \geq 1$ with $c_{i+k} = c_{i+k+f}$ for all $1 \leq k < j$ and $c_{i+j+f} > c_{i+j}$, and the situation is reversed in the case $i \notin J$ and $i+1 \in J$.

Note that if $e \geq p - 1$, then there is no restriction on $\psi_1|_{I_{F_p}}$; this is easy to see, because we may take $J = \emptyset$, and the x_i are then free to assume any value in $[0, e]$.

4.3. Necessary conditions: the irreducible case. We retain the notation and hypotheses of the previous subsection, but now we consider the case of an irreducible $\bar{\rho}$. In this case, $\bar{\rho}|_{G_L} \equiv \begin{pmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{pmatrix}$. Again, we compute the possibilities for

$\psi_1|_{I_{F_p}}$. Let \mathcal{H} be the finite flat group scheme with generic fibre descent data from K to L corresponding to $\bar{\rho}|_{G_L}$, and let \mathcal{G} be the subgroup scheme corresponding to ψ_1 . Note that the descent data on \mathcal{H} must extend to $\text{Gal}(K/F_p)$ while the descent data on \mathcal{G} must not.

As before, the Breuil module \mathcal{M} with descent data to L corresponding to \mathcal{G} may be written in the form

- $\mathcal{M} = ((k \otimes_{\mathbb{F}_p} k_E)[u]/u^{e(p^{2f}-1)}) \cdot v$,
- $(\text{Fil}^1 \mathcal{M})_i = u^{r_i} \mathcal{M}_i$,
- $\phi_1(u^{r_i} e_i v) = (1 \otimes \gamma_i) e_{i+1} v$ for some $\gamma_i \in k_E^\times$,
- $\hat{g}(\sum_{i=0}^{2f-1} e_i v) = \sum_{i=0}^{2f-1} (\bar{\eta}(g)^{k_i} \otimes 1) e_i v$ for all $g \in \text{Gal}(K/F_p)$.

Here the k_i , e_i are integers with $k_i \in [0, p^{2f} - 1]$ and $r_i \in [0, e(p^{2f} - 1)]$ satisfying $k_{i+1} \equiv p(k_i + r_i) \pmod{p^{2f} - 1}$. We may also write $\hat{g}(e_i v) = (1 \otimes \chi_i(g)) e_i v$ where $\chi_i = \sigma_i \circ \bar{\eta}^{k_i} \in \{\bar{\chi}, \bar{\chi}^q\}$.

We no longer know that $k_i = k_{i+f}$ (or equivalently that $\chi_i^q = \chi_{i+f}$), and in fact the following lemma shows that this is no longer the case.

Lemma 4.5. *We have $k_{i+1} \equiv p^f k_i \pmod{p^{2f} - 1}$, or equivalently $\chi_i = \chi_{i+f}$.*

Proof. Let \mathcal{N} be the Breuil module corresponding to \mathcal{H} . By Corollary 5.2 and the comment following it, the basis vector v of \mathcal{M} may be extended to a basis v, w of \mathcal{N} such that $\widehat{g}(e_i w) = (1 \otimes \chi_i^q(g))e_i w$.

Now we exploit the fact that the descent data on \mathcal{N} extends to descent data from K to $F_{\mathfrak{p}}$. Let $\varphi \in \text{Gal}(K/F_{\mathfrak{p}})$ be the element fixing ϖ and acting nontrivially on L , so that $\varphi^{-1}g\varphi = g^q$ for any $g \in \text{Gal}(K/L)$. Note that $\widehat{\varphi}(e_i) = e_{i+f}$. Let A_i be the invertible matrix with entries in $k_E[u]/(u^{e(p^{2f}-1)p})$ such that

$$\widehat{\varphi} \begin{pmatrix} e_i v \\ e_i w \end{pmatrix} = A_i \begin{pmatrix} e_{i+f} v \\ e_{i+f} w \end{pmatrix}.$$

The relation $g\varphi = \varphi g^q$ implies that

$$\begin{pmatrix} 1 \otimes \chi_i^q & 0 \\ 0 & 1 \otimes \chi_i \end{pmatrix} A_i = A_i \begin{pmatrix} 1 \otimes \chi_{i+f} & 0 \\ 0 & 1 \otimes \chi_{i+f}^q \end{pmatrix};$$

using the fact that $\chi_i \neq \chi_i^q$ we deduce that for each i , either $\chi_i^q = \chi_{i+f}$ and A_i is diagonal, or $\chi_i = \chi_{i+f}$ and A_i is skew-diagonal.

But suppose that A_i is diagonal for some i . Let a be the $(1, 1)$ -entry of A_i . Then

$$\widehat{\varphi}(e_{i+1}v) = (\widehat{\varphi} \circ \phi_1)(\gamma_i^{-1}u^{r_i}e_i v) = (\phi_1 \circ \widehat{\varphi})(\gamma_i^{-1}u^{r_i}e_i v) = \phi_1(\gamma_i^{-1}au^{r_i}e_{i+f}v)$$

is a multiple of $e_{i+f+1}v$, and A_{i+1} must also be diagonal; iterating, we see that A_i is diagonal for all i . But then $\widehat{\varphi}$ preserves \mathcal{M} , and the character ψ_1 of G_L extends to a character of G_F ; this contradicts the irreducibility of $\bar{\rho}$. Therefore A_i is skew-diagonal for all i , and $\chi_i = \chi_{i+f}$. \square

Remark 4.6. Though we shall not need this fact, with only slightly more work one can check that \mathcal{N} is actually the Breuil module direct sum of \mathcal{M} and the submodule \mathcal{M}' generated by w . Indeed, the commutativity of ϕ_1 and $\widehat{\varphi}$ implies that $\text{Fil}^1 \mathcal{N} \cap \mathcal{M}' = \sum_i u^{r_i+f} e_i \mathcal{M}'$ and $\phi_1(u^{r_i+f} e_i w)$ generates $e_{i+1} \mathcal{M}'$. Therefore \mathcal{M}' , equipped with the filtration and ϕ_1 induced from \mathcal{N} , is a Breuil module. Finally the map $\mathcal{M} \oplus \mathcal{M}' \hookrightarrow \mathcal{N}$ is a bijective morphism of Breuil modules; but bijective morphisms of Breuil modules are isomorphisms of Breuil modules.

Remark 4.7. We expect it to be true that $r_i + r_{i+f} = e(p^{2f} - 1)$, coming from the hypothesis that $\bar{\rho}$ lifts to a parallel potentially Barsotti-Tate representation (along with the decomposition $\mathcal{N} = \mathcal{M} \oplus \mathcal{M}'$). However, we will not actually need to prove this in order to determine the possibilities for ψ_1 , and so we temporarily ignore this expected condition.

Again, write

$$\bar{\chi} = \prod_{i=0}^{2f-1} \eta_i^{c_i}$$

with $0 \leq c_i \leq p-1$. Then we see that each k_i is congruent modulo $p^{2f} - 1$ to either

$$c_i + p^{2f-1}c_{i+1} + p^{2f-2}c_{i+2} + \cdots + pc_{i-1}$$

or

$$c_{i+f} + p^{2f-1}c_{i+f+1} + p^{2f-2}c_{i+f+2} + \cdots + pc_{i+f-1}.$$

We let J be the set of places such that the first possibility obtains. The fact that $k_{i+f} \equiv p^f k_i \pmod{p^{2f}-1}$ shows that $i \in J$ if and only if $i+f \in J$. As above, if $i \in J$ and $i+1 \in J$, or if $i \notin J$ and $i+1 \notin J$, we obtain $r_i \equiv 0 \pmod{p^{2f}-1}$. In either case, write $r_i = (p^{2f}-1)x_i$ for some $0 \leq x_i \leq e$. If $i \in J$ and $i+1 \notin J$, we see that

$$r_i = (p^{2f}-1)x_i + (p^f-1)(p^{f-1}(c_{i+f+1}-c_{i+1}) + p^{f-2}(c_{i+f+2}-c_{i+2}) + \cdots + (c_i - c_{i+f}))$$

for some x_i and that if $i \notin J$ and $i+1 \in J$, then

$$r_i = (p^{2f}-1)x_i + (p^f-1)(p^{f-1}(c_{i+1}-c_{i+f+1}) + p^{f-2}(c_{i+2}-c_{i+f+2}) + \cdots + (c_{i+f}-c_i)).$$

Again, we define

$$s_0 = \frac{p}{(p^{2f}-1)}(r_0 p^{2f-1} + r_1 p^{2f-2} + \cdots + r_{2f-1}),$$

so that $\phi_1|_{I_L} = \bar{\eta}_0^{k_0+s_0}$. Consider s_0 as a linear expression in the c_i . By the same reasoning as in the previous subsection it is easy to check that the coefficient of c_0 is always 0. Therefore the coefficient of c_0 in $k_0 + s_0$ is 1 if $0 \in J$, and p^f otherwise. By cyclic symmetry, the coefficient of c_i in $k_0 + s_0$ is p^{2f-i} if $i \in J$, and p^{f-i} if $i \notin J$. We thus see that

$$\psi_1|_{I_L} = \prod_{i \in J} \eta_i^{c_i} \prod_{i \notin J} \eta_i^{c_{i+f}} \prod_i \eta_i^{x_i}.$$

It remains to determine the possible values of the x_i . The initial analysis is just as in the last subsection. As already remarked, if $i \in J$ and $i+1 \in J$, or if $i \notin J$ and $i+1 \notin J$, then we may take any $0 \leq x_i \leq e$. Since the expression $(p^f-1)(p^{f-1}(c_{i+f+1}-c_{i+1}) + p^{f-2}(c_{i+f+2}-c_{i+2}) + \cdots + (c_i - c_{i+f}))$ is nonzero and is strictly between $1-p^{2f}$ and $p^{2f}-1$, in the other cases we allow either $0 \leq x_i \leq e-1$ or $1 \leq x_i \leq e$, depending on whether the sign of this expression is positive or negative. If $i \in J$ and $i+1 \notin J$, then the allowable range is $0 \leq x_i \leq e-1$ precisely when there is a $j \geq 1$ with $c_{i+k} = c_{i+k+f}$ for all $1 \leq k < j$ and $c_{i+j+f} > c_{i+j}$, and the situation is reversed in the case $i \notin J$ and $i+1 \in J$.

Finally, we consider restrictions on ψ_1 coming from the expectation that $r_i + r_{i+f} = e(p^{2f}-1)$, or equivalently that $x_i + x_{i+f} = e$. We will see that it suffices to consider restrictions coming from the determinant of $\bar{\rho}$: if $e < p-1$ then the determinant condition will actually imply that $x_i + x_{i+f} = e$, while if $e \geq p-1$ then we will see that all possibilities for $\psi_1|_{I_L}$ such that $\psi_1^{q+1}|_{I_L} = \det(\bar{\rho})|_{I_L}$ already occur among those ψ_1 with $x_i + x_{i+f} = e$ for all i (and so nothing extraneous is gained by forgetting the latter restriction).

Lemma 4.8. *If $\bar{\rho}$ has a parallel potentially Barsotti-Tate lift of type $\chi \oplus \chi^q$, then $\det \bar{\rho}|_{I_{\mathbb{F}_p}} = \epsilon \cdot \bar{\chi}^{q+1}$, where ϵ is the mod p cyclotomic character.*

Proof. This follows at once from Definition 3.2 and the results of section B.2 of [CDT99]. \square

Consider first the case $e < p-1$. By the Lemma, we must have $\psi_1^{q+1}|_{I_L} = \epsilon \cdot \bar{\chi}^{q+1}$. Writing $y_i = x_i + x_{i+f}$ this is easily seen to reduce to

$$\prod_{i=0}^{f-1} \omega_i^{y_i} = \epsilon = \omega_0^{e(p^f-1)/(p-1)}$$

or equivalently $p^{f-1}y_1 + p^{f-2}y_2 + \dots + y_f \equiv e(p^f - 1)/(p - 1) \pmod{p^f - 1}$. Note that the left hand side of this congruence lies in the interval $[0, 2e(p^f - 1)/(p - 1)]$, and since $e < p - 1$ the congruence must actually be an equality. Finally one checks easily that if each $y_i \in [0, 2e]$ and $p^{f-1}y_1 + p^{f-2}y_2 + \dots + y_f = e(p^f - 1)/(p - 1)$, then each y_i must actually equal e .

Next, suppose that $e \geq p - 1$. We let $J = \{0, \dots, 2f - 1\}$, and consider the possibilities for $\psi_1|_{I_L}$ that occur if we put $x_{i+f} = e - x_i$. If $X = p^{f-1}x_1 + \dots + x_f$ then $\psi_1|_{I_L} = \bar{\chi} \cdot \eta_f^{e(p^f-1)/(p-1)} \cdot \eta_0^{(1-p^f)X}$ and depends only on $X \pmod{p^f + 1}$. If $e \geq p$ then as x_1, \dots, x_f range over the interval $[0, e]$, the integer X ranges over an interval that includes $[0, p^f]$, and $\psi_1|_{I_L}$ ranges over all $p^f + 1$ characters with $\psi_1^{q+1}|_{I_L} = \epsilon \cdot \bar{\chi}^{q+1}$.

If instead $e = p - 1$, then X only ranges over the interval $[0, p^f - 1]$, and we obtain all possibilities for $\psi_1|_{I_L}$ satisfying the determinant condition except $\psi_1|_{I_L} = \bar{\chi}$. But performing the same analysis with $J = \emptyset$ gives us all possibilities for $\psi_1|_{I_L}$ satisfying the determinant condition except $\psi_1|_{I_L} = \bar{\chi}^q$; in particular since $\bar{\chi} \neq \bar{\chi}^q$ we obtain $\psi_1|_{I_L} = \bar{\chi}$ as a possibility with $J = \emptyset$.

Summarizing, if $\bar{\rho}$ is irreducible then the possibilities for $\psi_1|_{I_L}$ are as follows. If $e < p - 1$, then

$$\psi_1|_{I_L} = \prod_{i \in J} \eta_i^{c_i} \prod_{i \notin J} \eta_i^{c_i+f} \prod_i \eta_i^{x_i}$$

where $J \subset \{0, \dots, 2f - 1\}$ is a subset with $i + f \in J$ if and only if $i \in J$ and the x_i are nonnegative integers satisfying $x_i + x_{i+f} = e$; moreover if $i \in J$ and $i + 1 \notin J$ or vice-versa, then only one of $(x_i, x_{i+f}) = (0, e), (e, 0)$ may occur, depending on the sign of the next nonzero $c_{i+j+f} - c_{i+j}$. If $e \geq p - 1$, then the situation simplifies and $\psi_1|_{I_L}$ may be any character of niveau $2f$ satisfying $\psi_1^{q+1}|_{I_L} = \det(\bar{\rho})|_{I_L}$.

5. DESCENT DATA ON STRONGLY DIVISIBLE MODULES AND GALOIS TYPES

For this section only, let F/\mathbb{Q}_p be a finite extension. Suppose that K/F is a tamely ramified Galois extension with ramification index $e(K/F)$. Suppose moreover that there exists a uniformiser $\pi \in \mathcal{O}_K$ with $\pi^{e(K/F)} \in L$, where L is the maximal unramified extension of F contained in K ; then $K = L(\pi)$ and L contains all of the $e(K/F)$ th roots of unity. Let k denote the residue field of K (also equal to the residue field of L), and K_0 the maximal unramified extension of \mathbb{Q}_p contained in K .

Suppose that ρ is a potentially Barsotti-Tate representation $G_F \rightarrow \mathrm{GL}_n(E)$ that becomes Barsotti-Tate over K . We assume as usual that K_0 embeds into the coefficients E . Write $D := D_{\mathrm{st}, 2}^K(\rho)$ and let \mathcal{N} be a strongly divisible module with descent data over $S := S_{K, \mathcal{O}_E}$ contained in $S[1/p] \otimes_{K_0 \otimes \mathbb{Q}_p} E D$.

Let τ be the inertial Galois type of ρ . Note that τ factors through $I_L/I_K \cong \mathrm{Gal}(K/L)$. Since $\mathrm{Gal}(K/L)$ is abelian, τ decomposes as a direct sum of n characters $\chi_i : I_L \rightarrow \mathcal{O}_E^\times$, and we use the isomorphism $I_L/I_K \cong \mathrm{Gal}(K/L)$ to identify each χ_i as a character of $\mathrm{Gal}(K/L)$.

Proposition 5.1. *We have $\tau = \chi_1 \oplus \dots \oplus \chi_n$ if and only if there is an S -basis v_1, \dots, v_n of \mathcal{N} such that the descent data acts on \mathcal{N} via $\hat{g} \cdot v_i = (1 \otimes \chi_i(g))v_i$ for all $g \in \mathrm{Gal}(K/L)$.*

Proof. For each embedding $\sigma : K_0 \rightarrow E$, let e_σ denote the corresponding idempotent in $W(k) \otimes_{\mathbb{Z}_p} \mathcal{O}_E$, so that $S \cong \bigoplus_\sigma e_\sigma S$ with each $e_\sigma S$ a local domain. Since \widehat{g} fixes each e_σ , we see that \widehat{g} acts separately on each $e_\sigma \mathcal{N}$.

Suppose we know that \mathcal{N} has an S -basis v'_1, \dots, v'_n on which $\widehat{g} \cdot v'_i = \psi_i(g)v'_i$ for some characters $\psi_i : \text{Gal}(K/L) \rightarrow (W(k) \otimes_{\mathbb{Z}_p} \mathcal{O}_E)^\times$. The argument in the first three paragraphs of the proof of Proposition 6.6 of [GS] proves that D has a $K_0 \otimes_{\mathbb{Q}_p} E$ -basis on which \widehat{g} acts via the maps ψ_i . This is enough for the ‘if’ direction; for the ‘only if’ direction, recall that by definition of τ , we know that D also has a $K_0 \otimes_{\mathbb{Q}_p} E$ -basis on which \widehat{g} acts via the maps $1 \otimes \chi_i$. Since $K_0 \otimes_{\mathbb{Q}_p} E$ is not a domain it may not quite be the case that $\psi_i = 1 \otimes \chi_i$, but at least for each σ the multiset $\{e_\sigma \psi_i\}$ is equal to the multiset $\{\chi_i\}$: that is, we may relabel $e_\sigma v'_1, \dots, e_\sigma v'_n$ as $v_{1,\sigma}, \dots, v_{n,\sigma}$ in such a way that $\widehat{g} \cdot v_{i,\sigma} = \chi_i(g)v_{i,\sigma}$. If we define $v_i = \sum_\sigma v_{i,\sigma}$ then v_1, \dots, v_n is the desired basis. Thus we are reduced to the statement at the beginning of the paragraph.

In particular it is enough to show for each σ that the free $e_\sigma S$ -module $e_\sigma \mathcal{N}$ of rank n has a $e_\sigma S$ -basis on which \widehat{g} acts by characters $\text{Gal}(K/L) \rightarrow \mathcal{O}_E^\times$. Let $e_\sigma S_0$ be the subring of $e_\sigma S$ consisting of power series in $u^{e(K/F)}$. Observe that $e_\sigma S$ is free of rank $e(K/F)$ as an $e_\sigma S_0$ -module, with basis $1, \dots, u^{e(K/F)-1}$; this is because $e(K/F)$ divides the absolute ramification index of K , so that if p^α exactly divides u^{m-1} in S and a larger power of p divides u^m , then m is divisible by $e(K/F)$.

We now regard $e_\sigma \mathcal{N}$ as a free $e_\sigma S_0$ -module of rank $e(K/F)n$. Note that $\text{Gal}(K/L)$ acts on $e_\sigma S_0$ trivially, so that $e_\sigma \mathcal{N}$ is actually a $\text{Gal}(K/L)$ -representation over $e_\sigma S_0$. Since $\text{Gal}(K/L)$ is abelian and $p \nmid \#\text{Gal}(K/L)$, and since \mathcal{O}_E^\times contains the $e(K/F)$ th roots of unity, the module $e_\sigma \mathcal{N}$ actually has a simultaneous $e_\sigma S_0$ -basis of eigenvectors $y_1, \dots, y_{e(K/F)n}$ for the action of $\text{Gal}(K/L)$. Relabel the elements y_i so that y_1, \dots, y_n are a basis for the k_E -vector space $e_\sigma \mathcal{N}/(\mathfrak{m}_{e_\sigma S})e_\sigma \mathcal{N}$; here $\mathfrak{m}_{e_\sigma S}$ is the maximal ideal of $e_\sigma S$ and k_E is the residue field of E . By Nakayama’s lemma, the elements y_1, \dots, y_n are the desired $e_\sigma S$ -basis of $e_\sigma \mathcal{N}$. \square

The following corollary is immediate.

Corollary 5.2. *Let $\overline{\mathcal{N}}$ denote the Breuil module with descent data corresponding to the π_E -torsion in the Barsotti-Tate group corresponding to $\rho|_{G_K}$. If $\tau = \chi_1 \oplus \dots \oplus \chi_n$, then $\overline{\mathcal{N}}$ has a $(k \otimes k_E)[u]/(u^{e'})$ -basis $\overline{v}_1, \dots, \overline{v}_n$ such that $\widehat{g} \cdot \overline{v}_i = (1 \otimes \overline{\chi}_i(g))\overline{v}_i$ for all $g \in \text{Gal}(K/L)$. (Here e' is the absolute ramification index of K .)*

Let $e_i \in k \otimes k_E$ be one of our usual idempotents. Since the descent data fixes e_i , we see in particular that the descent data acts via $\overline{\chi}_1 \oplus \dots \oplus \overline{\chi}_n$ on each piece $e_i \overline{\mathcal{N}}$ of $\overline{\mathcal{N}}$.

6. LOCAL LIFTS

6.1. Lifts of certain rank two Breuil modules. We continue to use the following notation from Sections 4.2 and 4.3. Let L be the unramified quadratic extension of F_p , and K the splitting field of $u^{p^{2f}-1} - \pi_p$ over L . Let ϖ be a choice of $\pi_p^{1/(p^{2f}-1)}$ in K . Let k_0 and k denote the residue fields of F_p and K respectively, and if $g \in \text{Gal}(K/F_p)$ then we let $\overline{\eta}(g)$ be the image of $g(\varpi)/\varpi$ in k . Suppose that F_p has absolute ramification index e , and write $e' = (p^{2f} - 1)e$.

Let $E_0(u)$ be an Eisenstein polynomial for $\pi_{\mathfrak{p}}$, so that $E(u) = E_0(u^{p^{2f-1}})$ is an Eisenstein polynomial for ϖ . Write $E(u) = u^{e'} + pF(u)$; then $F(u)$ is a polynomial in $u^{p^{2f-1}}$ over $W(k_0)$ whose constant term is a unit.

Let E , a finite extension of \mathbb{Q}_p , denote the coefficient field for our representations, with integer ring \mathcal{O}_E and maximal ideal \mathfrak{m}_E . Enlarging E if necessary, we assume that E is ramified, that $W(k)$ embeds into E , and that a Galois closure of $F_{\mathfrak{p}}$ over \mathbb{Q}_p embeds into E . Let k_E denote the residue field of E . Write $S = S_{K, \mathcal{O}_E}$ (notation as in [Sav05, Sec. 4]). Recall that $\phi : S \rightarrow S$ is the $W(k)$ -semilinear, \mathcal{O}_E -linear map sending $u \mapsto u^p$. The group $\text{Gal}(K/F_{\mathfrak{p}})$ acts $W(k)$ -semilinearly on S via $g \cdot u = (\bar{\eta}(g) \otimes 1)u$. Set $c = \frac{1}{p}\phi(E(u)) \in S^\times$. Let $\varphi \in \text{Gal}(K/F_{\mathfrak{p}})$ denote the element fixing ϖ and acting nontrivially on L , so that $\varphi^{-1}g\varphi = g^q$ for $g \in \text{Gal}(K/L)$.

We now define a rank two Breuil module $\bar{\mathcal{N}}$ over $(k \otimes k_E)[u]/(u^{e'})$ with descent data from K to $F_{\mathfrak{p}}$ with generators \bar{v} and \bar{w} , as follows.

- $\text{Fil}^1 \bar{\mathcal{N}}$ is generated by $\sum_{i=0}^{2f-1} u^{r_i} e_i \bar{v}$ and $\sum_{i=0}^{2f-1} u^{r'_i} e_i \bar{w}$.
- $\phi_1(u^{r_i} e_i \bar{v}) = (1 \otimes \gamma_i) e_{i+1} \bar{v}$ and $\phi_1(u^{r'_i} e_i \bar{w}) = (1 \otimes \gamma'_i) e_{i+1} \bar{w}$.
- $\hat{g}(e_i \bar{v}) = (1 \otimes \chi_i(g))(e_i \bar{v})$ and $\hat{g}(e_i \bar{w}) = (1 \otimes \chi_i(g)^q)(e_i \bar{w})$ for $g \in \text{Gal}(K/L)$.

Here $\gamma_i, \gamma'_i \in k_E^\times$, the integers r_i, r'_i lie in the interval $[0, e']$, and $\chi_i \in \{\bar{\chi}, \bar{\chi}^q\}$ for each i . Let $J = \{i : \chi_i = \bar{\chi}\}$. We suppose furthermore that:

- we have $r_i + r'_i = e'$ for all i ,
- if $\chi_i = \chi_{i+1}$ then $p^{2f} - 1$ divides r_i and r'_i , and
- if $\chi_i \neq \chi_{i+1}$ then $\chi_i^q = (\sigma_i \circ \bar{\eta})^{r_i} \chi_i$.

One checks that the last two conditions are equivalent to the congruence conditions on the $r_i, r'_i \pmod{p^{2f-1}}$ that arise in Sections 4.2 and 4.3. For instance, if $i \in J$ and $i+1 \notin J$, then the condition that

$$r_i \equiv (p^f - 1)(p^{f-1}(c_{i+f+1} - c_{i+1}) + p^{f-2}(c_{i+f+2} - c_{i+2}) + \cdots + (c_i - c_{i+f}))$$

is equivalent to the condition that $\bar{\chi}^q = (\sigma_i \circ \bar{\eta})^{r_i} \bar{\chi}$. Finally, we assume that one of the following two sets of additional conditions holds:

$$\text{(RED)} \quad \begin{cases} r_i = r_{i+f} \text{ and } r'_i = r'_{i+f} \\ \gamma_i = \gamma_{i+f} \text{ and } \gamma'_i = \gamma'_{i+f} \\ \chi_{i+f} = \chi_i^q \\ \hat{\varphi}(\bar{v}) = \bar{v} \text{ and } \hat{\varphi}(\bar{w}) = \bar{w} \end{cases}$$

or

$$\text{(IRR)} \quad \begin{cases} r_i = r'_{i+f} \\ \gamma_i = \gamma'_{i+f} \\ \chi_{i+f} = \chi_i \\ \hat{\varphi}(\bar{v}) = \bar{w} \text{ and } \hat{\varphi}(\bar{w}) = \bar{v}. \end{cases}$$

Note that the relation $\hat{\varphi}^{-1} \circ \hat{g} \circ \hat{\varphi} = \hat{g}^p$ holds in either case, and so the module $\bar{\mathcal{N}}$ so-defined is indeed a Breuil module with descent data from K to $F_{\mathfrak{p}}$.

Theorem 6.1. *For each Breuil module $\bar{\mathcal{N}}$ as above, the generic fibre $\bar{\rho}$ of $\bar{\mathcal{N}}$ lifts to a parallel potentially Barsotti-Tate representation ρ with inertial type $\chi \oplus \chi^q$.*

Proof. We will show that $\bar{\mathcal{N}}$ lifts to a strongly divisible module \mathcal{N} with \mathcal{O}_E -coefficients and tame descent data from K to $F_{\mathfrak{p}}$ such that $\rho' := T_{\text{st}, 2}^{F_{\mathfrak{p}}}(\mathcal{N})[1/p]$ is a potentially Barsotti-Tate representation with inertial type $\chi \oplus \chi^q$ and all pairs

of labeled Hodge-Tate weights equal to $\{0, 1\}$. The representation ρ' need not be parallel, but since $\det(\rho')$ is the product of the cyclotomic character, a finite order character of order prime to p , and an unramified character, we may take ρ to be the twist of ρ' by a suitable unramified character with trivial reduction mod p .

Let \mathcal{N} be the free S -module generated by v and w ; as one would imagine, v, w will lift \bar{v}, \bar{w} respectively. Let $e_i \in W(k) \otimes_{\mathbb{Z}_p} \mathcal{O}_E$ also denote the idempotent lifting $e_i \in k \otimes_{\mathbb{F}_p} k_E$, and let $\tilde{\sigma}_i : K_0 \hookrightarrow E$ be the embedding that lifts σ_i . We define $\text{Fil}^1 \mathcal{N}$ to be the sum of $(\text{Fil}^1 S)\mathcal{N}$ and the submodules \mathcal{N}'_i of $e_i \mathcal{N}$ defined for each $0 \leq i < 2f$ as follows.

If $\chi_i = \chi_{i+1}$, let $g_i h_i$ be a monic factorization of $\tilde{\sigma}_i(E_0(u))$ in $\mathcal{O}_E[u]$ such that $\deg(g_i) = r_i/(p^{2f} - 1)$ and $\deg(h_i) = r'_i/(p^{2f} - 1)$. This is where we use our hypothesis that E contains all conjugates of $\pi_{\mathfrak{p}}$ over \mathbb{Q}_p . Take $G_i = g_i(u^{p^{2f}-1})$ and $H_i = h_i(u^{p^{2f}-1})$, so that $G_i H_i = \tilde{\sigma}_i(E(u))$, and let \mathcal{N}'_i be the S -module generated by $(1 \otimes G_i(u))e_i v$ and $(1 \otimes H_i(u))e_i w$.

If $\chi_i \neq \chi_{i+1}$ let $y_i z_i$ be a factorization of $-p\tilde{\sigma}_i(F(u))$ in $\mathcal{O}_E[u]$ such that $y_i \in \mathfrak{m}_E$ and $z_i \in \mathfrak{m}_E[u]$ or vice-versa. This is where we use our hypothesis that E is ramified. Take \mathcal{N}'_i to be the S -module generated by $e_i(u^{r_i}v + (1 \otimes y_i)w)$ and $e_i((1 \otimes z_i)v + u^{r'_i}w)$.

We impose the following extra conditions. If $\overline{\mathcal{N}}$ satisfies (RED), then we insist that $G_{i+f} = G_i$, $H_{i+f} = H_i$, $y_{i+f} = y_i$, and $z_{i+f} = z_i$; on the other hand if $\overline{\mathcal{N}}$ satisfies (IRR), then we require $G_i = H_{i+f}$ and $y_i = z_{i+f}$. Note that this is possible because $E(u)$ is defined over $W(k_0)$, so that $\tilde{\sigma}_{i+f}(E(u)) = \tilde{\sigma}_i(E(u))$.

We define descent data from K to $F_{\mathfrak{p}}$ on \mathcal{N} as follows, semilinearly with respect to the action of $\text{Gal}(K/F_{\mathfrak{p}})$ on S .

- If $i \in J$ and $g \in \text{Gal}(K/L)$, set $\hat{g}(e_i v) = (1 \otimes \chi(g))e_i v$ and $\hat{g}(e_i w) = (1 \otimes \chi(g)^q)e_i w$.
- If $i \notin J$ and $g \in \text{Gal}(K/L)$, set $\hat{g}(e_i v) = (1 \otimes \chi(g)^q)e_i v$ and $\hat{g}(e_i w) = (1 \otimes \chi(g))e_i w$.
- If $\overline{\mathcal{N}}$ satisfies (RED) then set $\hat{\varphi}(v) = v$ and $\hat{\varphi}(w) = w$.
- If $\overline{\mathcal{N}}$ satisfies (IRR) then set $\hat{\varphi}(v) = w$ and $\hat{\varphi}(w) = v$.

In either of the last two cases, using the fact that φ acts trivially on \mathcal{O}_E and takes $e_i \mapsto e_{i+f}$, one checks that $\hat{\varphi}^{-1} \hat{g} \hat{\varphi} = \hat{g}^q$, so that this descent data extends to $\text{Gal}(K/F_{\mathfrak{p}})$ in a well-defined way. One checks with little difficulty from the definition of $\text{Fil}^1 \mathcal{N}$ and the conditions on $\overline{\mathcal{N}}$ that this descent data preserves $\text{Fil}^1 \mathcal{N}$.

Finally we wish to define a map $\phi : \mathcal{N} \rightarrow \mathcal{N}$, semilinear with respect to ϕ on S and such that $\phi_1 = \frac{1}{p} \phi|_{\text{Fil}^1 \mathcal{N}}$ is well-defined and satisfies

$$(6.1.1) \quad \phi_1((1 \otimes G_i(u))e_i v) = \tilde{\gamma}_i e_{i+1} v$$

$$(6.1.2) \quad \phi_1((1 \otimes H_i(u))e_i w) = \tilde{\gamma}'_i e_{i+1} w$$

if $\chi_i = \chi_{i+1}$ and

$$(6.1.3) \quad \phi_1(e_i(u^{r_i}v + (1 \otimes y_i)w)) = \tilde{\gamma}_i e_{i+1} v$$

$$(6.1.4) \quad \phi_1(e_i((1 \otimes z_i)v + u^{r'_i}w)) = \tilde{\gamma}'_i e_{i+1} w$$

if $\chi_i \neq \chi_{i+1}$. Here $\tilde{\gamma}_i, \tilde{\gamma}'_i$ are lifts of γ_i, γ'_i to \mathcal{O}_E^\times that satisfy $\tilde{\gamma}_{i+f} = \tilde{\gamma}_i$ and $\tilde{\gamma}'_{i+f} = \tilde{\gamma}'_i$ in case (RED) and $\tilde{\gamma}'_{i+f} = \tilde{\gamma}_i$ in case (IRR).

If $\chi_i = \chi_{i+1}$ we may satisfy (6.1.1) and (6.1.2) by setting

$$\begin{aligned}\phi(e_i v) &= c^{-1} \phi(1 \otimes H_i(u)) \tilde{\gamma}_i e_{i+1} v \\ \phi(e_i w) &= c^{-1} \phi(1 \otimes G_i(u)) \tilde{\gamma}'_i e_{i+1} w.\end{aligned}$$

If $\chi_i \neq \chi_{i+1}$, then since

$$(E(u) \otimes 1) e_i v = u^{r'_i} (e_i (u^{r_i} v + (1 \otimes y_i) w)) - y_i (e_i ((1 \otimes z_i) v + u^{r'_i} w))$$

and similarly for $(E(u) \otimes 1) e_i w$, we should set

$$\begin{aligned}\phi(e_i v) &= c^{-1} e_{i+1} (u^{pr'_i} \tilde{\gamma}_i v - \phi(y_i) \tilde{\gamma}'_i w) \\ \phi(e_i w) &= c^{-1} e_{i+1} (u^{pr_i} \tilde{\gamma}'_i w - \phi(z_i) \tilde{\gamma}_i v).\end{aligned}$$

Extending this map additively ϕ -semilinearly to all of \mathcal{N} , one checks that equations (6.1.1)–(6.1.4) hold, so that $\phi(\text{Fil}^1 \mathcal{N})$ is contained in $p\mathcal{N}$ and generates it over S . One checks furthermore that ϕ commutes with the descent data on \mathcal{N} that was constructed in preceding paragraphs.

It is now evident that $(\mathcal{N}, \text{Fil}^1 \mathcal{N}, \phi)$ with the given descent data is a lift of $\overline{\mathcal{N}}$. It remains to check that \mathcal{N} satisfies the rest of the axioms of a strongly divisible module with coefficients and descent data (namely, conditions (2), (5)–(8), and (12) of [Sav05, Def. 4.1]) and to prove our claims about the representation ρ' .

To check that $\text{Fil}^1 \mathcal{N} \cap I\mathcal{N} = I\text{Fil}^1 \mathcal{N}$ for an ideal $I \subset \mathcal{O}_E$, observe that it suffices to check separately for each i that $e_i \text{Fil}^1 \mathcal{N} \cap e_i I\mathcal{N} = e_i I\text{Fil}^1 \mathcal{N}$. If $\chi_i \neq \chi_{i+1}$ then this follows by exactly the same argument as in the proof of [GS, Thm 6.5] (the algebraic claim being made is literally identical). If $\chi_i = \chi_{i+1}$ then the argument is even easier. Each coset in $e_i(\text{Fil}^1 \mathcal{N}/(\text{Fil}^1 S)\mathcal{N})$ has a representative of the form $e_i(aG_i v + bH_i w)$ with $a, b \in \mathcal{O}_E[u]$ such that $\deg(a) < \deg(H_i)$ and $\deg(b) < \deg(G_i)$: terms of higher degree can be absorbed into $(\text{Fil}^1 S)\mathcal{N}$ by using the relation $E(u) \otimes 1 = 1 \otimes G_i H_i$ in $e_i S$. If $aG_i e_i v + bH_i e_i w + s_1 v + s_2 w$ lies in $e_i I\mathcal{N}$ (with $s_1, s_2 \in e_i \text{Fil}^1 S$) then $aG_i e_i + s_1$ must lie in $e_i IS$; the same must be true of $aG_i e_i$ and s_1 individually since they have no terms in common of the same degree in their unique expansions of the form $\sum_{j \geq 0} q_j(u) E(u)^j / j!$ with $\deg(q_j) < \deg(E(u))$. Then since G_i is monic the coefficients of a must lie in I . Similarly $s_2 \in e_i IS$ and the coefficients of b lie in I . It follows that $aG_i e_i v + bH_i e_i w + s_1 v + s_2 w$ actually lies in $e_i I\text{Fil}^1 \mathcal{N}$.

As for the axioms (5)–(8) and (12) of [Sav05, Def. 4.1] concerning the monodromy operator N , again we appeal to arguments in the proof of [GS, Thm 6.5]: ignoring the action of \mathcal{O}_E and the descent data and regarding $(\mathcal{N}, \text{Fil}^1 \mathcal{N}, \phi)$ simply as a strongly divisible \mathbb{Z}_p -module over K , it follows from [Bre00, Prop 5.1.3(1)] that there exists a *unique* $W(k) \otimes \mathbb{Z}_p$ -endomorphism $N : \mathcal{N} \rightarrow \mathcal{N}$ satisfying axioms (5)–(8) of [Sav05, Def. 4.1], except that we have axiom (5) only with respect to $s \in S_{K, \mathbb{Z}_p}$ until we know that N commutes with the action of \mathcal{O}_E . This commutativity, as well as the commutativity between N and the descent data (axiom (12)), follows by uniqueness of the operator N . This completes the proof that \mathcal{N} is a strongly divisible module.

As before, set $\rho' = T_{\text{st}, 2}^{F_p}(\mathcal{N})[1/p]$, the potentially Barsotti-Tate Galois representation associated to \mathcal{N} , and let $D := D_{\text{st}, 2}^K(\rho')$. The claim that the Galois type of ρ' is $\chi \oplus \chi^q$ follows directly by Proposition 5.1 applied with respect to the S -basis $v' = \sum_{i \in J} e_i v + \sum_{i \notin J} e_i w$ and $w' = \sum_{i \notin J} e_i v + \sum_{i \in J} e_i w$ of \mathcal{N} .

The last thing to verify is that all pairs of labeled Hodge-Tate weights of ρ' are $\{0, 1\}$. Recall that if we regard $K \otimes_{\mathbb{Q}_p} E$ as an $S[1/p]$ -algebra via the map $u \mapsto \varpi \otimes 1$, then by [Bre97, Prop. 6.2.2.1] there is an isomorphism

$$f_{\varpi} : (K \otimes_{\mathbb{Q}_p} E) \otimes_{S[1/p]} \mathcal{N}[1/p] \cong D_K := K \otimes_{W(k)[1/p]} D$$

that identifies the filtrations on both sides. It follows that D_K is the free $(K \otimes_{\mathbb{Q}_p} E)$ -module generated by $f_{\varpi}(v)$ and $f_{\varpi}(w)$, and we need to show that $\text{Fil}^1 D_K$ is a free submodule of rank one in D_K . It suffices to check for each i that $e_i \text{Fil}^1 D_K$ is a free $e_i(K \otimes_{\mathbb{Q}_p} E)$ -submodule of rank one in $e_i D_K$. If $\chi_i \neq \chi_{i+1}$ then this follows as in the last paragraph of the proof of [GS, Prop 6.6]: the images of $e_i(u^{r_i}v + y_i w)$ and $e_i(z_i v + u^{r'_i} w)$ under f_{ϖ} are scalar multiples of one another, and each generates a free submodule of rank one in $e_i \text{Fil}^1 D_K$.

For the case $\chi_i = \chi_{i+1}$, we note that each of our idempotents $e_i \in K \otimes_{\mathbb{Q}_p} E$ decomposes as a sum of idempotents e_{τ} , where τ ranges over the $e(p^{2f} - 1)$ embeddings $K \hookrightarrow E$ extending $\tilde{\sigma}_i$. Since $\varpi \otimes 1 = 1 \otimes \tau(\varpi)$ in $e_{\tau}(K \otimes_{\mathbb{Q}_p} E)$, we deduce that the e_{τ} -component of $f_{\varpi}(G_i(u)e_i v)$ is nonzero precisely for those τ such that the root $\tau(\varpi)$ of $\tilde{\sigma}_i(E(u))$ is not a root of $G_i(u)$, and similarly the e_{τ} -component of $f_{\varpi}(H_i(u)e_i w)$ is nonzero for those τ such that $\tau(\varpi)$ is not a root of $H_i(u)$. It follows that $e_i \text{Fil}^1 D_K$ is free of rank one, generated by the image of $e_i(G_i(u)v + H_i(u)w)$ under f_{ϖ} . \square

7. AN EXPLICIT DESCRIPTION OF THE SET OF WEIGHTS

We maintain the notation of the previous three sections. The results of sections 4 and 6 may be combined to give a complete description of when a semisimple 2-dimensional mod p representation of G_{F_p} admits a parallel potentially Barsotti-Tate lift of type $\chi \oplus \chi^q$ with $\chi \neq \chi^q$. In turn, this furnishes an explicit description of the set $W^2(\bar{\rho})$ in the case that the restriction of $\bar{\rho}$ to each decomposition group above p is semisimple.

Theorem 7.1. *Write $\bar{\chi} = \prod_{i=0}^{2f-1} \eta_i^{c_i}$, with $0 \leq c_i \leq p-1$. Assume that $\bar{\rho}|_{G_{F_p}}$ is semisimple. Then $\bar{\rho}|_{G_{F_p}}$ has a parallel potentially Barsotti-Tate lift of type $\chi \oplus \chi^q$ if and only if one of the following three possibilities holds.*

- (1) $e \geq p-1$ and $\det \bar{\rho}|_{I_{F_p}} = \epsilon \cdot \bar{\chi}^{q+1}$.
- (2) $\bar{\rho}|_{G_{F_p}} \cong \begin{pmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{pmatrix}$ is decomposable, with $(\psi_1 \psi_2)|_{I_{F_p}} = \epsilon \cdot \bar{\chi}^{q+1}$, and

$$\psi_1|_{I_{F_p}} = \prod_{i \in J} \omega_{\pi(i)}^{c_i} \prod_{i=0}^{f-1} \omega_i^{x_i}$$

where $J \subset \{0, \dots, 2f-1\}$ is a subset with $i+f \in J$ if and only if $i \notin J$, where π is the natural projection from $\mathbb{Z}/2f\mathbb{Z}$ to $\mathbb{Z}/f\mathbb{Z}$, and where the x_i satisfy the following conditions:

- If $i \in J$ and $i+1 \in J$, or if $i \notin J$ and $i+1 \notin J$, then we may take any $0 \leq x_i \leq e$.
- In the other cases we allow either $0 \leq x_i \leq e-1$ or $1 \leq x_i \leq e$. If $i \in J$ and $i+1 \notin J$, then the allowable range is $0 \leq x_i \leq e-1$ precisely when there exists $j \geq 1$ with $c_{i+k} = c_{i+k+f}$ for all $1 \leq k < j$ and $c_{i+j+f} > c_{i+j}$, and the situation is reversed in the case $i \notin J$ and $i+1 \in J$.

(3) $\bar{\rho}|_{G_{F_p}}$ is irreducible, and $\bar{\rho}|_{I_{F_p}} \cong \begin{pmatrix} \psi & 0 \\ 0 & \psi^q \end{pmatrix}$ with

$$\psi = \prod_{i \in J} \eta_i^{c_i} \prod_{i \notin J} \eta_i^{c_{i+f}} \prod_i \eta_i^{x_i}$$

where $J \subset \{0, \dots, 2f-1\}$ is a subset with $i+f \in J$ if and only if $i \in J$ and the x_i are nonnegative integers satisfying $x_i + x_{i+f} = e$, in addition to the following condition:

- If $i \in J$ and $i+1 \notin J$, then we allow $0 \leq x_i \leq e-1$ precisely when there is a $j \geq 1$ with $c_{i+k} = c_{i+k+f}$ for all $1 \leq k < j$ and $c_{i+j+f} > c_{i+j}$, and we otherwise allow $1 \leq x_i \leq e$. The situation is reversed in the case $i \notin J$ and $i+1 \in J$.

Note that by the discussion at the end of section 4.3, the condition in (3) that the x_i are nonnegative integers satisfying $x_i + x_{i+f} = e$ may be replaced by the condition that $0 \leq x_i \leq e$ for all i and $\psi^{q+1}|_{I_{F_p}} = \epsilon \cdot \bar{\chi}^{q+1}$.

Proof. The necessity of these conditions follows from Lemma 4.8 and the discussions of section 4.2 and 4.3. For their sufficiency, consider first the case that $\bar{\rho}|_{G_{F_p}}$ is irreducible. Then by the discussion of section 4.3, along with the conditions given in (3), there is a Breuil module $\bar{\mathcal{N}}$ as in section 6 so that the generic fibre of $\bar{\mathcal{N}}$ restricted to I_{F_p} is $\bar{\rho}|_{I_{F_p}}$. Then, since we are in the irreducible case, the generic fibre of $\bar{\mathcal{N}}$ is an unramified twist of $\bar{\rho}|_{G_{F_p}}$, and the representation coming from Theorem 6.1 applied to $\bar{\mathcal{N}}$ is an unramified twist of the desired lift of $\bar{\rho}|_{G_{F_p}}$.

In the case that $\bar{\rho}|_{G_{F_p}}$ is reducible, note that case (1) will follow immediately from case (2) applied with $J = \emptyset$. For case (2), observe that by the discussion in section 4.2, the generic fibre of the rank one Breuil module \mathcal{M} of section 4.2 agrees with ψ_1 up to an unramified twist; but in fact by Corollary 4.3 the parameters γ_i may be chosen so that these characters agree on the whole group G_{F_p} . Then the Breuil module \mathcal{M} from section 4.2 may be extended to a Breuil module $\bar{\mathcal{N}}$ as in section 6, satisfying the conditions (RED), whose generic fibre is $\bar{\rho}|_{G_{F_p}}$. The result again follows from Theorem 6.1. \square

Theorem 7.2. *Suppose that for each $v|p$, $\bar{\rho}|_{G_{F_v}}$ is semisimple. Let $\sigma = \otimes_{v|p} \sigma_v$ be a weight. Then $\sigma \in W^2(\rho)$ if and only if for each $v|p$ we have*

- (1) σ_v is of type I, and the conditions of Theorem 7.1 apply with $\mathfrak{p} = v$ and $\chi = \tilde{\sigma}_v$ (regarded as a character of I_{F_v} by local class field theory), or
- (2) σ_v is of type II, and

$$\bar{\rho}|_{I_{F_v}} \cong \sigma_v \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}$$

where σ_v is regarded as a character of I_{F_v} via local class field theory.

In particular, if for each $v|p$ the ramification index of F_v is at least $p-1$, and σ_v is of type I for each $v|p$, then $\sigma \in W^2(\bar{\rho})$ if and only if for each $v|p$ we have $\det \bar{\rho}|_{G_{F_p}} = \epsilon \cdot \sigma_v^{q+1}$.

Proof. This all follows immediately from Definition 3.5 and Theorem 7.1, except for the case that σ_v is of type II. In this case, the necessity of the given condition follows from Lemma 3.6, and the sufficiency is straightforward: twisting reduces

to the case $\sigma_v = 1$, when the result follows from the existence of a non-crystalline extension of the trivial character by the cyclotomic character. \square

8. PROOF OF THE WEIGHT CONJECTURE

Recall that we are assuming that F is a totally real field. We now prove in many cases that $W(\bar{\rho}) = W^?(\bar{\rho})$, by combining the results of earlier sections with the lifting machinery of Khare-Wintenberger, as interpreted by Kisin. In particular, we use the following result.

Proposition 8.1. *Suppose that $p > 2$ and that $\bar{\rho} : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is modular. Assume that $\bar{\rho}|_{G_{F(\zeta_p)}}$ is irreducible. If $p = 5$ and the projective image of $\bar{\rho}$ is isomorphic to $\mathrm{PGL}_2(\mathbb{F}_5)$, assume further that $[F(\zeta_p) : F] = 4$. Suppose that for each place $v|p$, θ_v is an inertial type for I_{F_v} such that $\bar{\rho}|_{G_{F_v}}$ has a non-ordinary parallel potentially Barsotti-Tate lift of type θ_v . Then $\bar{\rho}$ has a modular lift which is parallel potentially Barsotti-Tate of type θ_v for all $v|p$.*

Proof. This is a special case of Corollary 3.1.7 of [Gee06a]. \square

Recall that we defined the set of weights $W^?(\bar{\rho})$ conjecturally associated to $\bar{\rho}$ in section 3, and that in the case that the restrictions of $\bar{\rho}$ to decomposition groups above p are semisimple, Theorem 7.2 gives an explicit description of $W^?(\bar{\rho})$.

Theorem 8.2. *Suppose that $p > 2$ and that $\bar{\rho} : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is modular. Assume that $\bar{\rho}|_{G_{F(\zeta_p)}}$ is irreducible. If $p = 5$ and the projective image of $\bar{\rho}$ is isomorphic to $\mathrm{PGL}_2(\mathbb{F}_5)$, assume further that $[F(\zeta_p) : F] = 4$. Then $W^?(\bar{\rho}) \subset W(\bar{\rho})$. If $\sigma \in W^?(\bar{\rho})$, and $\sigma = \otimes_{v|p} \sigma_v$ with each σ_v of type I, then $\sigma \in W(\bar{\rho})$. In particular, if there are no places $v|p$ for which $\bar{\rho}|_{G_{F_v}}$ is a twist of an extension of the trivial character by the cyclotomic character, then $W(\bar{\rho}) = W^?(\bar{\rho})$.*

Proof. The inclusion $W^?(\bar{\rho}) \subset W(\bar{\rho})$ already follows from Lemma 3.4. The rest of the result follows from Lemma 3.4, Proposition 8.1 and Lemma 3.6, because if σ_v is of type I, any lift of type τ_v only becomes crystalline over a nonabelian extension, and is thus certainly non-ordinary. \square

Remark 8.3. It should be possible to improve this result to prove the equality $W^?(\bar{\rho}) = W(\bar{\rho})$ under the assumption that $\bar{\rho}$ has a modular lift of parallel weight two which is ordinary at any place v such that there is an element of $W^?(\bar{\rho}|_{G_{F_v}})$ of type II. This would involve strengthening Proposition 8.1 to include potentially semistable lifts, and the use of $R = T$ theorems for Hida families. The required results are not in the literature in the appropriate level of generality, however.

APPENDIX A. CORRIGENDUM TO [Sav05]

The second author wishes to take this opportunity to correct an error in [Sav05], as a consequence of which there is one more family of strongly divisible modules that must be studied by the methods of [Sav05]. Once this is done, the remaining claims of [Sav05] are unaffected. We adopt the notation of [Sav05] without further comment, and all numbered references are to that paper.

The mistake is in the statement and proof of Theorem 6.12(4). In the situation of that item, if $m = 1 + (p + 1)j$ — i.e., if $i = 1$ — then the two characters ω_2^{m+p} and ω_2^{pm+1} are both characters of niveau one, and are equal; hence in this case the

proof of Theorem 6.12(4) does *not* show that $T_{st,2}^{\mathbb{Q}_p}(\mathcal{M}/\mathfrak{m}_E)|_{I_p}$ decomposes as a sum of two conjugate characters. In fact, for each choice c of square root of \bar{w} , the map $\mathcal{M}'_2 \rightarrow \mathcal{M}_E(F_2/\mathbb{Q}_{p^2}, e_2, c, m-1)$ extends to a map $\mathcal{M}' \rightarrow \mathcal{M}_E(F_2/\mathbb{Q}_p, e_2, c, j)$; by Proposition 5.4(1), we conclude when $i = 1$ that

$$T_{st,2}^{\mathbb{Q}_p}(\mathcal{M}/\mathfrak{m}_E)|_{I_p} \otimes_{\mathbf{k}_E} \bar{\mathbb{F}}_p \cong \lambda_{c^{-1}}\omega^{1+j} \oplus \lambda_{-c^{-1}}\omega^{1+j}.$$

This means that when $i = 1$ and $\text{val}(b) > 0$ we still need to construct a strongly divisible lattice in $\mathcal{D}_{m,[1:b]}$ whose reduction mod p has trivial endomorphisms; or, conversely, we need to study deformations of type $\tilde{\omega}_2^m \oplus \tilde{\omega}_2^{pm}$ (with $i = 1$) of non-split residual representations of the form

$$\begin{pmatrix} \lambda_{c^{-1}}\omega^{1+j} & * \\ 0 & \lambda_{-c^{-1}}\omega^{1+j} \end{pmatrix}.$$

We rectify this omission now. Our statements are numbered to mesh with the original article if one drops the **A.** prefix.

Lemma A.6.7. (2) *If $i = 1$, $\text{val}_p(b) > 0$, and w is a square in E , then there is $X \in S_{F_2, \mathcal{O}_E}^\times$ satisfying*

$$X(1 \otimes wb) = 1 \otimes w - \left(1 + \frac{u^{pe_2}}{p}\right) X\phi(X).$$

Proof. The constant term of X may be taken to be $1 \otimes x_0$ where x_0 is either root of $x_0^2 + wb x_0 - w$ in \mathcal{O}_E^\times . The recursion for the coefficient x_n of u^n is $x_n(x_0 + wb) =$ lower terms, and so the recursion can be solved to obtain $X \in S_{F_2, \mathcal{O}_E}^\times$. \square

Moreover, since $\text{val}_p(b) > 0$, by putting the variable B for b we obtain an element X_B of $S_{F_2, \mathcal{O}_E[[B]]}$ which specializes to X under the map $\mathcal{O}_E[[B]] \rightarrow \mathcal{O}_E$ sending $B \mapsto b$. Note that the image of X in $(\mathbb{F}_{p^2} \otimes \mathbf{k}_E)[u]/u^{e_2 p}$ is $1 \otimes c$ with c a square root of \bar{w} . Assume henceforth that the coefficient field E contains a square root of w . Now Proposition 6.10 is modified as follows.

Proposition A.6.10. *In the case $i = 1$ and $\text{val}_p(b) > 0$, we instead define*

$$\mathcal{M}_{m,[1:b]} = S_{F_2, \mathcal{O}_E} \cdot g_1 + S_{F_2, \mathcal{O}_E} \cdot g_2$$

$$\begin{aligned} g_1 &= \mathbf{e}_1 + \frac{X}{pw} u^{p(p-1)} \mathbf{e}_2 \\ g_2 &= \mathbf{e}_2, \end{aligned}$$

and this is a strongly divisible \mathcal{O}_E -module with descent data inside $\mathcal{D}_{m,[1:b]}$.

Proof. Put $\mathcal{M} = \mathcal{M}_{m,[1:b]}$. Observe that $h := u^{p-1}g_1 + \left(\frac{X}{w} + (1 \otimes b)\right)g_2$ lies in $\text{Fil}^1\mathcal{M}$. Since $\frac{X}{w} + (1 \otimes b)$ is a unit in S_{F_2, \mathcal{O}_E} and g_1 does not lie in $\text{Fil}^1\mathcal{M}$, we deduce that $\text{Fil}^1\mathcal{M} = S_{F_2, \mathcal{O}_E} \cdot h + (\text{Fil}^1 S_{F_2, \mathcal{O}_E})\mathcal{M}$. From this it is easy to check that $I\mathcal{M} \cap \text{Fil}^1\mathcal{M} = I\text{Fil}^1\mathcal{M}$. Finally, we compute that

$$\begin{aligned} \phi(g_1) &= \phi(X)u^{p^2(p-1)}g_1 + \left(1 - X\phi(X)\frac{u^{pe_2}}{pw}\right)g_2 \\ \phi(g_2) &= pwg_1 - Xu^{p(p-1)}g_2 \end{aligned}$$

both lie in \mathcal{M} ; using the defining relation for X we find $\phi_1(h) = (1 \otimes w)X^{-1}g_1 \in \mathcal{M}$ and conclude that \mathcal{M} is a strongly divisible module. \square

Now amend Theorem 6.12(4) so that it applies only to the case $i > 1$, and add the following.

Theorem A.6.12. (5) *If $i = 1$ and $\text{val}_p(b) > 0$, then $T_{\text{st},2}^{\mathbb{Q}_p}(\mathcal{M}/\mathfrak{m}_E)$ is independent of b and*

$$T_{\text{st},2}^{\mathbb{Q}_p}(\mathcal{M}/\mathfrak{m}_E) \cong \begin{pmatrix} \lambda_{-c^{-1}\omega^{1+j}} & * \\ 0 & \lambda_{c^{-1}\omega^{1+j}} \end{pmatrix}$$

with $* \neq 0$.

Proof. Write $\mathcal{M}' = T_0(\mathcal{M}/\mathfrak{m}_E)$. Then $\text{Fil}^1\mathcal{M}'$ is generated by $u^{p-1}g_1 + c^{-1}g_2$ and $u^{e_2}g_1$, with $\phi_1(u^{p-1}g_1 + c^{-1}g_2) = cg_1$ and $\phi_1(u^{e_2}g_1) = u^{p^2(p-1)}cg_1 + g_2$. Note that $\phi_1(u^{p(p-1)}g_2) = -cg_2$. There is evidently a nontrivial map $\mathcal{M}' \rightarrow \mathcal{M}_E(F_2/\mathbb{Q}_p, e_2, c, j)$ sending $g_2 \mapsto 0$ and $g_1 \mapsto u^{p^2}\mathbf{e}$. On the other hand if $f : \mathcal{M}' \rightarrow \mathcal{M}_E(F_2/\mathbb{Q}_p, e_2, d, n)$ is a nontrivial map sending $g_1 \mapsto \alpha\mathbf{e}$ and $g_2 \mapsto \beta\mathbf{e}$, then α, β must both be polynomials in u^p since g_1, g_2 are in the image of ϕ_1 . Now if $\beta \neq 0$ then the relation $f \circ \phi_1 = \phi_1 \circ f$ on $u^{p(p-1)}g_2$ implies that β is a unit times u^p ; but then $f(u^{p-1}g_1 + c^{-1}g_2) \in \langle u^{e_2}\mathbf{e} \rangle$ implies that α has a linear term, a contradiction. Therefore $\beta = 0$, and then it is easy to check that $c = d$ and $j = n$. It follows that $* \neq 0$. \square

(We also note the following typos in the published version of the proof of Theorem 6.12(4): in the first sentence, the expression $\phi_1(u^{e_2})$ should be $\phi_1(u^{e_2}g_2)$; in the last sentence, the characters λ_c should both be $\lambda_{c^{-1}}$.)

The proof of Corollary 6.15(2) should then invoke Theorem 6.12(5) in lieu of Theorem 6.12(3) in the case of representations ρ to which Theorem 6.12(5) applies, noting that the two choices for x_0 lead to different reductions of ρ .

We now turn to deformation spaces of strongly divisible modules. The proof of the following proposition is identical to the proof that the corresponding module $\mathcal{M}_{m,[1:b]}$ of Proposition 6.10 is a strongly divisible module. As noted in Remark 6.20, we omit the description of N in the strongly divisible module below.

Proposition A.6.21. *There exists a strongly divisible module with descent data and $\mathcal{O}_E[[B]]$ -coefficients as follows.*

(6) *If $i = 1$ and assuming that w is a square in E ,*

$$\begin{aligned} \mathcal{M}_X &= (S_{F_2, \mathcal{O}_E[[B]]}) \cdot g_1 \oplus (S_{F_2, \mathcal{O}_E[[B]]}) \cdot g_2, \\ \text{Fil}^1\mathcal{M}_X &= S_{F_2, \mathcal{O}_E[[B]]} \cdot (u^{p-1}g_1 + (w^{-1}X_B + (1 \otimes B))g_2) + (\text{Fil}^1 S_{F_2, \mathcal{O}_E[[B]]})\mathcal{M}_X, \\ \phi(g_1) &= \phi(X_B)u^{p^2(p-1)}g_1 + \left(1 - X_B\phi(X_B)\frac{u^{pe_2}}{pw}\right)g_2, \\ \phi(g_2) &= pwg_1 - X_Bu^{p(p-1)}g_2, \\ \widehat{g}(g_1) &= (\widetilde{\omega}_2^m \otimes 1)g_1, \quad \widehat{g}(g_2) = (\widetilde{\omega}_2^{pm} \otimes 1)g_2. \end{aligned}$$

Finally, one must amend the proof of Theorem 6.24 to include a proof that the canonical injection

$$R(2, \tau(\mathcal{M}_X), \bar{\rho}(\mathcal{M}_X))_{\mathcal{O}_E} \rightarrow \mathcal{O}_E[[B]]$$

is a surjection; this proceeds exactly along the strategy outlined in the proof of Theorem 6.24. Indeed, let \mathcal{M}'' denote the minimal Breuil module with descent data from F_2 to \mathbb{Q}_p associated to the character $\lambda_{-c^{-1}\omega^{1+j}}$, with generator h such that $\phi_1(h) = -c^{-1}h$. Then a map $f : \mathcal{M}'' \rightarrow T_0(\mathcal{M}_X/(\mathfrak{m}_E, B^2))$ must send h to an

element of the form $\alpha u^{e_2} g_1 + \beta(u^{p-1} g_1 + (w^{-1} X_B + B) g_2)$ (where, abusing notation, we identify elements of $S_{F_2, \mathcal{O}_E[[B]]}$ with their images in $(\mathbb{F}_{p^2} \otimes \mathbf{k}_E[B]/(B^2))[u]/u^{e_2 p}$). Write $\alpha = \alpha_0 + B\alpha_1$ and $\beta = \beta_0 + B\beta_1$ to separate out the terms involving B . The relation $f(\phi_1(h)) = \phi_1(f(h))$ shows first that $\alpha_0 = au^p$, $\beta_0 = -au^{p-2}$ for some $a \in \mathbf{k}_E$, by considering the relation mod B ; then, after some algebra, the full relation eventually implies $a = 0$. Thus the image of f lies in $B \cdot T_0(\mathcal{M}_X/(\mathfrak{m}_E, B^2))$, as desired.

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