Associated Primes and Primary Decomposition

David Savitt

March 14, 2000

These notes are an attempt to unify the presentations of associated primes and primary decomposition given by Atiyah-Macdonald [1], Bourbaki [2], and Matsumura [3]. Their definitions differ somewhat, and we hope that the ways in which we have chosen to highlight these differences will lead to the reader having a greater understanding of the important concepts described herein.

The differing definitions all coincide in the case of finitely generated modules over Noetherian rings; and indeed this is almost exclusively the context in which associated primes arise, so there isn't much to worry about. The most likely place you'll run across these objects in a more general context is in EGA, which uses definitions consistent with Bourbaki.

In what follows, all rings are commutative rings with identity.

1 Basics

We begin with:

Definition 1.1. Let A be a ring and M an A-module. If P is a prime of A, we say that P is an *associated prime* of M if there exists $x \in M$ such that $\text{ann}(x) = P$. Let $\text{Ass}_A(M)$, or simply $\text{Ass}(M)$, denote the primes associated to M.

However, we must immediately issue:

Warning 1.2. Atiyah-MacDonald's definition of associated prime in [1] is different from the definition in [2] and [3]. Atiyah and MacDonald define the associated primes of a module M to be the primes which occur as the *radical* of $\text{ann}(x)$ for some $x \in M$. To keep our terminology clear, we will call such primes AM-associated, and we will denote the set of AM-associated primes of M by $\text{Ass}^{\text{AM}}(M)$.

Example 1.3. To see that the two definitions are actually different, let $A = k[x_1, x_2, \ldots]$ be the power series ring in countably many variables, and let $I = (x_1^2, x_2^2, \ldots)$ be the ideal generated by the squares of the variables. Then I is the annihilator of $1 \in A/I$, and $\text{rad}(I) = (x_1, x_2, \ldots)$ is prime but is not the annihilator of any element of A/I.

We will develop, in parallel, the theory of both associated primes and AM-associated primes.

Observe that if $\text{ann}(x) = P$, then the submodule $Ax \subset M$ is isomorphic to A/P. Conversely, if M has a submodule isomorphic to A/P, then any nonzero element of that submodule has annihilator P; so $P \in \text{Ass}(M)$ (and hence is in $\text{Ass}^{\text{AM}}(M)$ as well). For example, it easily follows that:

Example 1.4. $\text{Ass}(A/P) = \text{Ass}^{\text{AM}}(A/P) = \{P\}$.

If I is an ideal of A, we will often say "associated primes of I" to refer to the associated primes of A/I; this should not cause any confusion.

Notice that there's no reason, a priori, that an A-module M should have any associated (resp., AM-associated) primes at all, since the annihilator (resp., the radical of the annihilator) of an arbitrary element of M need not be prime. However, we do have the following proposition:

Proposition 1.5. *Let $F = \{\mathrm{ann}(x) \mid x \in M,\ x \neq 0\}$, partially ordered under inclusion. Then any maximal element of F is a prime ideal, and hence is an associated prime (and AM-associated) of M.*

We leave the proof of this as an easy exercise; or, see Theorem 6.1 in [3]. Now if A is a Noetherian ring, then any nonempty collection of ideals contains maximal elements, and hence we conclude:

Corollary 1.6. *If A is a Noetherian ring and M is a nonzero A-module, then M has at least one associated prime.*

So already we begin to see that the theory of associated primes works more smoothly in the case where A is Noetherian: we're at least guaranteed that associated primes always exist! In fact, there's another very good reason to prefer the situation when A is Noetherian – in that case, the notion of associated prime and AM-associated prime coincide:

Proposition 1.7. *Every associated prime is AM-associated. If A is Noetherian, then every AM-associated prime is associated.*

Proof: The first statement of the proposition is clear. Conversely, suppose that P is an AM-associated prime of M. We will soon see in proposition 2.1 that in the case where A is Noetherian, to check whether P is an associated prime of M it suffices to check that PA_P is an associated prime of M_P. Furthermore, also by proposition 2.1, if P is an AM-associated prime of M, then PA_P is an AM-associated prime of M_P. Without loss of generality we may therefore assume that A is a local ring and P is its maximal ideal. Since P is AM-associated to M, there exists an element $x \in M$ such that $\mathrm{rad}(\mathrm{ann}(x)) = P$; as A is Noetherian, it follows that $P^n \subset \mathrm{ann}(x)$ for some positive integer n. Again using the fact that A is Noetherian, it follows from proposition 1.5 that there exists $y \in M$ whose annihilator P' contains $\mathrm{ann}(x)$ and is an associated prime of M. From $P^n \subset P'$ it follows that $P \subset P'$, and by maximality of P we conclude that $P = P'$ is an associated prime of M. □

Finally, we conclude this section with a useful fact and one corollary of it:

Lemma 1.8. *If A is a ring (not necessarily Noetherian) and if*

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence of A-modules, then

$$\mathrm{Ass}(M) \subset \mathrm{Ass}(M') \cup \mathrm{Ass}(M'').$$

The same statement is true with Ass *replaced by* $\mathrm{Ass}^{\mathrm{AM}}$.

Proof: Suppose $P \in \mathrm{Ass}(M)$, so that M has a submodule Ax which is isomorphic to A/P. Recall that any nonzero element of Ax has annihilator equal to P; if then $Ax \cap M' \neq \{0\}$, we conclude that M' contains an element whose annihilator is P, and so $P \in \mathrm{Ass}(M')$. On the other hand, if $Ax \cap M' = \{0\}$, it follows that Ax injects into M'' under the map $M \to M''$; so M'' has a submodule isomorphic to A/P, and $P \in \mathrm{Ass}(M'')$.

The proof in the case of AM-associated primes is slightly more complicated. If $P \in \mathrm{Ass}^{\mathrm{AM}}(M)$, select $x \in M$ such that $\mathrm{ann}(x) = I$ with $\mathrm{rad}(I) = P$. If $y = ax \in Ax$ with $a \notin P$, then $\mathrm{ann}(y) = \{b \mid ba \in \mathrm{ann}(x)\} \subset P$ (using the fact that $a \notin P$ and $\mathrm{rad}(I) = P$ to conclude $b \in P$). Consequently, if there exists $a \in A - P$ with $ax \in M'$, we may conclude that $P \in \mathrm{Ass}^{\mathrm{AM}}(M')$. On the other hand, suppose $ax \in M' \cap Ax$ implies $a \in P$. Letting x'' denote the image of x in M'', it follows that $\mathrm{ann}(x) \subset \mathrm{ann}(x'') \subset P$, and again $P \in \mathrm{Ass}^{\mathrm{AM}}(M'')$. □

Corollary 1.9. *If A is Noetherian and M is a finitely generated A-module, then M has only finitely many associated primes.*

Proof: Let $M_0 = 0$. Observe that if M is nonzero, then it has an associated prime, and so we may select a submodule M_1 of M such that $M_1 \cong A/P_1$ for some prime ideal P_1. Repeating this procedure inductively, we produce an increasing chain of submodules $M_0 \subset M_1 \subset M_2 \cdots$ such that $M_i/M_{i-1} \cong A/P_i$ for some associated prime $P_i \in \mathrm{Ass}(M/M_{i-1})$. Since M is finitely generated over a Noetherian ring A, we know that M is a Noetherian module, and hence this increasing chain must terminate at some $M_n = M$. By a repeated application of our lemma, we see that

$$\mathrm{Ass}(M) \subset \cup_{i=1}^{n} \mathrm{Ass}(A/P_i).$$

But we know from our example that $\mathrm{Ass}(A/P_i) = \{P_i\}$, and so $\mathrm{Ass}(M) \subset \{P_1, \cdots P_n\}$. □

2 Associated primes and localisation

We briefly turn to studying the effect of localisation on the associated primes of a module.

Proposition 2.1. *Let S be a multiplicative subset of the ring A. Regard $\mathrm{Spec}(A_S)$ as a subset of $\mathrm{Spec}(A)$ in the obvious fashion.*

1. *If M is an A_S-module, then $\mathrm{Ass}_A(M) = \mathrm{Ass}_{A_S}(M)$.*
2. *If M is an A-module, then $\mathrm{Ass}_A(M) \cap \mathrm{Spec}(A_S) \subset \mathrm{Ass}_A(M_S)$. Furthermore, equality holds if A is Noetherian.*
3. *The above remains true if Ass is replaced by $\mathrm{Ass}^{\mathrm{AM}}$.*

Proof: To prove (1), suppose M is an A_S-module and $P \in \mathrm{Ass}_A(M)$ (respectively, ${\mathrm{Ass}^{\mathrm{AM}}}_A(M)$). Then there exists an element $x \in M$ with $\mathrm{ann}_A(x) = P$ (resp., $\mathrm{rad}(\mathrm{ann}_A(x)) = P$). Since $x \neq 0$, it follows that $P \cap S = \emptyset$. Now for $a \in A$ and $s \in x$, $(a/s)x = 0$ implies $atx = 0$ for some $t \in S$. However, $at \in \mathrm{ann}_A(x) \subset P$ and $t \notin P$ implies $a \in P$, and thus $\mathrm{ann}_{A_S}(x) \subset PA_S$. Since $P = \mathrm{ann}_A(x) \subset \mathrm{ann}_{A_S}(x)$ (resp. $P = \mathrm{rad}(\mathrm{ann}_A(x)) \subset \mathrm{rad}(\mathrm{ann}_{A_S}(x))$), we obtain $PA_S = \mathrm{ann}_{A_S}(x)$ and $PA_S \in \mathrm{Ass}_{A_S}(M)$ (resp. $PA_S = \mathrm{rad}(\mathrm{ann}_{A_S}(x))$ and $PA_S \in {\mathrm{Ass}^{\mathrm{AM}}}_{A_S}(M)$.

The reverse inclusion is clear, using the facts that $\mathrm{ann}_A(x) = A \cap \mathrm{ann}_{A_S}(x)$, that $PA_S \cap A = P$ if $P \cap S = \emptyset$, and, in the AM-associated case, that $\mathrm{rad}_A(I \cap A) = \mathrm{rad}_{A_S}(I) \cap A$ for I an ideal of A_S.

To prove (2), suppose $P \in \mathrm{Ass}_A(M) \cap \mathrm{Spec}(A_S)$, i.e. that we have $x \in M$ with $\mathrm{ann}(x) = P$ and $P \cap S = \emptyset$. If $(a/s)x = 0$ in M_S, then $atx = 0$ in M for some $t \in S$, and since $t \notin P$ it follows that $a \in P$. Therefore $\mathrm{ann}_{A_S}(x) = PA_S$ in M_S, as desired. Just as for (1), the proof in the AM-associated case is entirely analogous.

Finally, we prove equality in (2) in the Noetherian case (so, to our relief, we need not distinguish between associated and AM-associated primes – however, to avoid circular reasoning in proposition 1.7, we are careful to use only the properties of associated primes in our proof). If $PA_S \in \mathrm{Ass}(M_S)$, choose $x \in M_S$ with $\mathrm{ann}(x) = PA_S$. Take $s \in S$ such that $y = sx \in M$. If $ay = 0$ in M, it certainly follows that $a \in P$. However, it might well be that $a \in P$ with $ay \neq 0$ in M: indeed, we only know that $ay = 0$ in M_S, i.e. that there exists $t_a \in S$ so that $at_a y = 0$ in M. However, if P is finitely generated, we can choose generators $a_1, \ldots, a_k$ for P and replace y by $z = t_{a_1} \cdots t_{a_k} y$, and then indeed $\mathrm{ann}_A(z) = P$, completing the proof. □

Recall that $\mathrm{Supp}(M)$, the support of M, is defined to be $\{P \in \mathrm{Spec}(A) \mid M_P \neq 0\}$. We use the preceeding result to obtain:

Theorem 2.2. *If A is a ring and M is an A-module, then $\mathrm{Ass}(M) \subset \mathrm{Ass}^{\mathrm{AM}}(M) \subset \mathrm{Supp}(M)$. If A is Noetherian, then the minimal elements of all three sets coincide.*

Proof: To prove the first assertion, suppose $P \in \mathrm{Ass}^{\mathrm{AM}}(M)$, i.e. that M has a submodule isomorphic to A/I for some ideal I with $\mathrm{rad}(I) = P$. Consider the exact sequence

$$0 \to A/I \to M$$

defining that submodule. Applying the exact functor $- \otimes_A A_P$, we obtain an exact sequence

$$0 \to A_P/IA_P \to M_P.$$

Since $IA_P \subset PA_P \subsetneq A_P$, M_P must be nonzero.

To prove the second assertion, first convince yourself that it suffices to prove: if $P \in \mathrm{Supp}(M)$, then P contains a prime in $\mathrm{Ass}(M)$. However, if $P \in \mathrm{Supp}(M)$, then $M_P \neq 0$ and since A (and thus A_P) is Noetherian we know that $\mathrm{Ass}(M_P) \neq \emptyset$. By the equality statement in part (2) of 2.1, we find that $\mathrm{Ass}(M) \cap \mathrm{Spec}(A_P) \neq \emptyset$, which is precisely the statement that $\mathrm{Ass}(M)$ has a prime contained in P. □

These minimal primes are called the *isolated* primes of M; the non-minimal elements of $\mathrm{Ass}(M)$ are called embedded primes.

Example 2.3. Let $A = k[x, y]$ and $I = (x^2, xy)$. Then the associate primes of I are (x) and (x, y). The ideal (x) is an isolated prime, and (x, y) is embedded.

Example 2.4. Let I be an ideal in the polynomial ring $k[x_1, \ldots, x_n]$. Then the isolated primes of I correspond to the irreducible components of the variety cut out by the polynomials in I.

3 Primary decomposition

Definition 3.1. Let $N \subset M$ be A-modules. We say that N is a primary submodule of M if $x \neq N$ and $ax \in N$ imply together that $a^\nu M \subset N$ for some positive integer ν. This is equivalent to the statement that if a is a zero-divisor for M/N, then $a \in \mathrm{rad}(\mathrm{ann}(M/N))$.

Warning 3.2. The definition of primary submodule in [2] differs from the definition in [1] and [3]: Bourbaki only defines a primary submodule $N \subset M$ in the case when A is Noetherian, and Bourbaki defines N to be primary if and only if for any zero divisor a of M/N and $x \in M$, there exists ν_x such that $a^{\nu_x} x \in M$. We call such submodules "primary-B". The two definitions evidently coincide if M is finitely generated. In what follows, we will not treat questions about primary-B submodules; instead, we leave it as an exercise to the reader to figure out how to rework the following theorems and proofs for primary-B submodules.

Example 3.3. If $I \subset A$ is a primary submodule, we say that I is a primary ideal. I is a primary ideal if and only if $ab \in I$ implies that either $a \in I$ or $b^\nu \in I$ for some ν sufficiently large; or, equivalently, that all zero-divisors of A/I are nilpotent. One observes immediately that the radical of a primary ideal is a prime ideal. See pp. 50-51 of [1] for additional basic properties of primary ideals.

The first fundamental result is:

Theorem 3.4. *Let A be a ring, and $N \subset M$ be A-modules.*

1. *If N is primary, then* $\mathrm{ann}(M/N)$ *is a primary ideal,* $P = \mathrm{rad}(\mathrm{ann}(M/N))$ *is prime, and* $\mathrm{Ass}^{\mathrm{AM}}(M/N) = \{P\}$. *(In this case say that N is P-primary.)*
2. *If N is P-primary and A is Noetherian, then* $\mathrm{Ass}(M/N) = \{P\}$.

3. *Conversely, if A is Noetherian,* $\mathrm{rad}(\mathrm{ann}(M/N)) = P$*, and* $\mathrm{Ass}(M/N) = \{P\}$*, then N is P-primary.*

Proof:

1. Suppose $ab \in \mathrm{ann}(M/N)$. Either $a \in \mathrm{ann}(M/N)$ or else $ax \notin N$ for some $x \in M$. In the latter case, since $abx \in N$, the fact that N is primary implies that $b^\nu M \subset N$ for some ν, and therefore $b^\nu \in \mathrm{ann}(M/N)$, proving that $\mathrm{ann}(M/N)$ is a primary ideal. Let $P = \mathrm{rad}(\mathrm{ann}(M/N))$.

   Suppose $Q \in \mathrm{Ass}^{\mathrm{AM}}(M/N)$, so that $\mathrm{rad}(\mathrm{ann} x) = Q$ for some $x \in M/N$. Plainly $\mathrm{ann}(M/N) \subset \mathrm{ann}(x)$, so $P = \mathrm{rad}(\mathrm{ann}(M/N)) \subset \mathrm{rad}(\mathrm{ann}(x)) = Q$. On the other hand, if $a \in Q$ we know that $a^\mu x = 0$ for some μ, and hence from the very definition of primary submodules we know that $a \in \mathrm{rad}(\mathrm{ann}(M/N)) = P$. Therefore $P = Q$, and indeed $\mathrm{Ass}^{\mathrm{AM}}(M/N) \subset \{P\}$.

   Next, if $x \in M/N$ is nonzero and $ax = 0$, from the fact that N is primary we know $a \in \mathrm{rad}(\mathrm{ann}(M/N)) = P$. Consequently we find that $\mathrm{ann}(x)$ is squeezed:

   $$\mathrm{ann}(M/N) \subset \mathrm{ann}(x) \subset P.$$

   Taking radicals throughout this equation yields $\mathrm{rad}(\mathrm{ann}(x)) = P$, so $P \in \mathrm{Ass}^{\mathrm{AM}}(M/N)$.

2. Clear by (1), since Ass and $\mathrm{Ass}^{\mathrm{AM}}$ coincide in the Noetherian case. (Alternately, if A is Noetherian, we know that M/N has at least one associated prime, and using (1) we conclude that $\mathrm{Ass}(M/N) = \{P\}$.)

3. If a is a zero-divisor of M/N we know by proposition 1.5 that $a \in P$, hence $a \in \mathrm{rad}(\mathrm{ann}(M/N))$ and N is primary. □

Example 3.5. The conditions of (3) are satisfied if A is Noetherian, M/N is finitely generated, and $\mathrm{Ass}(M/N) = \{P\}$, and so in this case we conclude N is P-primary. To see this claim, observe that if A is Noetherian, by the proof of 2.2 we know that $P \in \mathrm{Supp}(M/N)$ and every prime of $\mathrm{Supp}(M/N)$ contains P. Furthermore, since M/N is finitely generated, it is easy to see that $\mathrm{Supp}(M/N) = \{Q \in \mathrm{Spec}(A) \mid Q \supset \mathrm{ann}(M/N)\}$, and so $\mathrm{rad}(\mathrm{ann}(M/N)) = P$.

Example 3.6. By the preceeding example, it follows that if A is a Noetherian ring, an ideal I is P-primary if and only if $\mathrm{Ass}(I) = \{P\}$.

Example 3.7. Let A and I be as in example 1.3. Then I is an example of a primary ideal such that $\mathrm{Ass}(A/I)$ is empty.

Example 3.8. If $N', N'' \subset M$ are P-primary A-modules, then so is $N' \cap N''$. For if $a \in A$ and $x \in M$ with $x \notin N' \cap N''$ and $ax \in N' \cap N''$, without loss of generality suppose $x \notin N'$. Since $ax \in N'$, it follows that $a \in \mathrm{rad}(\mathrm{ann}(M/N')) = P = \mathrm{rad}((\mathrm{ann}(M/N''))$. Therefore, for some ν sufficiently large we have both $a^\nu M \subset N'$ and $a^\nu M \subset N''$, and $a \in \mathrm{rad}(\mathrm{ann}(M/(N' \cap N''))$. Hence $N' \cap N''$ is primary. The above argument also shows that $\mathrm{rad}(\mathrm{ann}(M/(N' \cap N''))) \subset P$ (as any nonzero element of the left-hand side is a zero-divisor $M/(N' \cap N'')$), and since the reverse inclusion is obviously true, we find that $N' \cap N''$ is actually P-primary.

Before moving on, we prove a proposition on the behaviour of primary modules under localisation:

Proposition 3.9. *Let A be a ring, $N \subset M$ a P-primary submodule, and S a multiplicative subset of A.*

1. *If $S \cap P \neq \emptyset$, then $N_S = M_S$.*
2. *If $S \cap P = \emptyset$, then N_S is a PA_S-primary submodule of M_S, and $N_S \cap M = N$.*

Proof:

1. Suppose $a \in S \cap P$. Sinec $a \in P = \mathrm{rad}(\mathrm{ann}(M/N))$, it follows that for any $x \in M$ there exists ν such that $a^\nu x = y \in N$. Consequently $x = y/a^\nu \in N_S$, as desired.

2. Suppose a/s is a zero-divisor for M_S/N_S, i.e. $a \in A$, $x \in M$, $s, t \in S$, and $(a/s)(x/t) \in N_S$ with $x/t \notin N_S$. Then $x \notin N$ and $uax \in N$ for some $u \in S$. Therefore ua is a zero-divisor for M/N, and so resides in $\mathrm{rad}(\mathrm{ann}(M/N)) = P$; since $u \notin P$, we conclude $a \in P$ and $a/s \in PA_S$. One readily checks that $PA_S \subset \mathrm{rad}(\mathrm{ann}(M_S/N_S))$, which completes the proof that N_S is primary.

   To see that N_S is actually PA_S-primary, we need to check that $\mathrm{rad}(\mathrm{ann}(M_S/N_S)) \subset PA_S$ or, equivalently, that $\mathrm{ann}(M_S/N_S) \subset PA_S$. However, if $a/s \in \mathrm{ann}(M_S/N_S)$, observe that for any $x \in M$ we find $x(a/s) \in N_S$, i.e. $atx \in N$ for some $t \in S$. In particular, taking $x \notin N$ we find that at is a zero-divisor for M/N, and so $at \in P$. Since $t \notin P$, we conclude $a \in P$, as desired.

   Finally, we verify that $N_S \cap M = N$. But this is clear: we know that $N_S \cap M = \{x \in M \mid sx \in N \text{ for some } s \in S\}$. If $sx \in N$ with $x \notin M$, we would conclude s is a zero-divisor for M/N and hence $s \in P$; but $P \cap S = \emptyset$, and so $sx \in N$ must imply $x \in N$. □

We end this section with a definition:

Definition 3.10. If $N \subset M$ are A-modules, then a *primary decomposition* for N is an expression $N = N_1 \cap \cdots \cap N_k$ such that each N_i is a primary A-submodule of M. A primary definition of N in which N_i is P_i-primary is said to be irredundant if (1) $P_i \neq P_j$ if $i \neq j$ and (2) no N_i can be omitted from the intersection $N_1 \cap \cdots \cap N_k$ to yield a shorter primary decomposition for N.

Using example 3.8, we observe that any primary decomposition may be reduced to an irredundant primary decomposition.

3.1 Existence of Primary Decompositions

In this subsection we prove the main result on the existence of primary decompositions, recalling first:

Definition 3.11. Given A-modules $N \subset M$, the submodule N is said to be irreducible if for any submodules $N', N'' \subset M$ satisfying $N' \cap N'' = N$, we either have $N' = N$ or $N'' = N$.

Theorem 3.12. *Let A be a Noetherian ring and M a finitely generated A-module. Then any irreducible submodule of M is a primary submodule, and hence any submodule $N \subset M$ has a primary decomposition.*

Proof: To prove the first assertion, suppose that $N \subset M$ is a submodule which is not primary; we will prove that it is not irreducible. Since M is finitely generated, by example 3.5 we learn that $\mathrm{Ass}(M/N)$ contains (at least) two distinct primes P_1 and P_2. Then M/N has submodules isomorphic to A/P_1 and A/P_2, whose intersection in M/N must be 0 (considering, for example, the annihilators of elements in the submodules). Hence 0 is not irreducible in M/N, and we conclude that N is not irreducible in M.

For the second assertion, we simply note that since M is Noetherian, N automatically has an irreducible decomposition, which by the first assertion is a primary decomposition. □

3.2 Uniqueness of Primary Decompositions

In this subsection, we suppose that A is a ring and $N \subset M$ is a submodule with irredundant primary decomposition $N = N_1 \cap \cdots \cap N_k$, where N_i is P_i-primary. (We say that N is decomposable.) We investigate what we can say about the uniqueness of this decomposition. The first important result is:

Theorem 3.13. $\mathrm{Ass}^{\mathrm{AM}}(M/N) = \{P_1, \ldots, P_k\}$.

Proof: From the natural injection

$$M/N \hookrightarrow M/N_1 \oplus \cdots \oplus M/N_k$$

it is clear that $\mathrm{Ass}^{\mathrm{AM}}(M/N) \subset \{P_1, \ldots, P_k\}$. To see the reverse direction, suppose that $x \notin N_i$ and $x \in N_1 \cap \cdots \cap N_{i-1} \cap N_{i+1} \cap \cdots N_k$. (Such an x exists by the irredundancy of our given primary decomposition.) Because $x \in N_j$ for all $j \neq i$, we obtain

$$\{a \in A \mid ax \in N\} = \cap_j \{a \in A \mid ax \in N_j\} = \{a \in A \mid ax \in N_i\},$$

and hence

$$\mathrm{ann}(x_N) = \mathrm{ann}(x_{N_i})$$

where x_N, x_{N_i} are the images of x in M/N and M/N_i respectively. The final paragraph of the proof of (1) in theorem 3.4 proves that $\mathrm{rad}(\mathrm{ann}(x_{N_i})) = P_i$, hence $\mathrm{rad}(\mathrm{ann}(x_N)) = P_i$ and $P_i \in \mathrm{Ass}^{\mathrm{AM}}(M/N)$. □

Corollary 3.14. *The isolated primes of M/N are precisely the primes which are minimal among those which contain* $\mathrm{ann}(M/N)$. *(Cf. theorem 2.2.)*

Proof: We know that

$$\mathrm{ann}(M/N) = \cap_i \mathrm{ann}(M/N_i),$$

so taking radicals yields

$$\mathrm{rad}(\mathrm{ann}(M/N)) = \cap_i P_i.$$

The primes which are minimal among those containing $\mathrm{ann}(M/N)$ are exactly the primes which are minimal among those containing the left-hand side of the above equality; while the isolated primes of M/N are precisely the primes which are minimal among those containing the right-hand side. (Recall that if P, P_i are prime, $P \supset \cap P_i$ implies $P \supset P_i$ for some i.) □

Example 3.15. Let A and I be as in example 1.3. Then I is primary, so is its own primary decomposition. This provides an example of a module N with primary decomposition such that $\mathrm{Ass}(M/N) \subsetneq \mathrm{Ass}^{\mathrm{AM}}(M/N)$.

Our primary decomposition for N will provide a primary decomposition for any localisation of N.

Proposition 3.16. *Suppose $N = N_1 \cap \cdots \cap N_k \subset M$ is a primary decomposition with N_k a P_k-primary submodule of M, and suppose S is a multiplicative subset of A. If $N_1, \ldots, N_k$ are ordered such that $P_i \cap S = \emptyset$ precisely when $i \leq j$ for $j \geq 1$, then*

$$N_S = (N_1)_S \cap \cdots \cap (N_k)_S$$

is a primary decomposition for N_S.

Proof: This is evident, using proposition 3.9 and the fact that the localisation of an intersection is the intersection of the localisations. □

Corollary 3.17. *If $N \subset M$ has a primary decomposition,*

$$\mathrm{Ass}^{\mathrm{AM}}(M/N) \cap \mathrm{Spec}(A_S) = \mathrm{Ass}^{\mathrm{AM}}((M/N)_S).$$

In particular, note the case when $N = 0$.

(Cf. part (2) of proposition 2.1.) This follows from theorem 3.13 and the proof of the above proposition.

The final result of this section is:

Theorem 3.18. *If P_i is a minimal element of $\{P_1, \cdots, P_k\}$ (under inclusion), then N_i is uniquely determined (i.e. the P_i-primary component in the decomposition is the same in any irredundant primary decomposition of N): in fact, $N_i = N_{P_i} \cap M$.*

Proof: Since P_i is minimal in $\mathrm{Ass}^{\mathrm{AM}}(N)$ and our primary decomposition is irredundant, it follows that for any $j \neq i$ the intersection $P_j \cap (A - P_i) \neq \emptyset$. Using the preceeding proposition, we immediately see that

$$N_{P_i} = (N_i)_{P_i}.$$

By the final statement in part (2) of proposition 3.9, we find that

$$N_i = M \cap (N_i)_{P_i} = M \cap N_{P_i},$$

completing the proof. □

Example 3.19. In the above theorem, the assumption that P_i is minimal is crucial. For example, the ideal $(x^2, xy) \subset k[x, y]$ has the following two primary decompositions: $(x) \cap (x^2, xy, y^2)$ and $(x) \cap (x^2, y)$. The (x)-primary component is the same in each, but the (x, y)-primary components are different.

4 Questions and Exercises

4.1 Questions

1. What is an example of a module M over a (non-Noetherian) ring A such that $\mathrm{Ass}^{\mathrm{AM}}(M) = \emptyset$? Is there such an example?

2. Can you weaken the hypotheses on theorem 3.4, part (3)?

3. Are there any mistakes or typos in the above exposition? (If so, please notify me!)

4.2 Exercises

1. Prove proposition 1.5. (For that matter, prove everything in these notes that I stated without proof.)

2. Give an example to show that equality need not hold in lemma 1.8.

3. If M_1, M_2 are submodules of M such that $M = M_1 + M_2$, is $\mathrm{Ass}(M) = \mathrm{Ass}(M_1) \cup \mathrm{Ass}(M_2)$? How about $\mathrm{Ass}^{\mathrm{AM}}$?

4. What happens in the previous problem if the sum $M = M_1 + M_2$ is a *direct* sum?

5. If A is Noetherian and M is an A-module, prove that the set of zero-divisors for M is equal to the union of all the associated primes of M.

6. Prove the same result as in the above problem, replacing the hypothesis that A is Noetherian with the hypothesis that 0 is a decomposable submodule of M, and replacing "associated primes" with "AM-associated primes".

7. If A is Noetherian and I is an ideal of A, show that I is primary if and only if A/I has no embedded primes.

8. If I is an ideal of A, prove that: (i) if $\mathrm{rad}(I)$ is a maximal ideal, then I is primary; (ii) the powers of a maximal ideal are primary. Furthermore, provide examples of ideals I and Noetherian rings A such that: (iii) I is primary but is not the power of a prime ideal; (iv) I is is the power of a prime ideal but is not primary; (v) $\mathrm{rad}(I)$ is a prime ideal but I is not primary.

9. Let M be a finitely generated module over a Noetherian ring A. Show that the following are equivalent: (i) M is of finite length; (ii) every $P \in \mathrm{Ass}(M)$ is a maximal ideal of A; (iii) every $P \in \mathrm{Supp}(M)$ is a maximal ideal of A.

10. Let M be a finitely generated module over a Noetherian ring A, and let P be a prime of A. Show that M_P is an A_P-module of finite length if and only if P is a minimal element of $\mathrm{Ass}(M)$.

11. Let M be an A-module and S a subset of $\mathrm{Ass}(M)$. Show that there exists a submodule N of M such that $\mathrm{Ass}(N) = \mathrm{Ass}(M) - S$ and $\mathrm{Ass}(M/N) = S$. Prove the same, with Ass replaced with $\mathrm{Ass}^{\mathrm{AM}}$. (Hint: consider the collection of submodules whose associated primes are all contained in $\mathrm{Ass}(M) - S$.)

12. Rewrite section 3, everywhere replacing "primary" with "primary-B". How do our theorems and proofs have to change? What more can be said. (For example: in part (1) of theorem 3.4, it's no longer true that $\mathrm{rad}(\mathrm{ann}(M/N))$ is prime – with what should we replace that object? In part (3) of the same theorem, the assumption that $\mathrm{rad}(\mathrm{ann}(M/N)) = P$ may simply be omitted. And if A is Noetherian and $\{M_i\}$ is an arbitrary family of P-primary-B submodules of M, the intersection $\cap_i M_i$ is P-primary-B.)

13. Let A be a Noetherian ring, M a finitely generated A-module and N any A-module. Compute $\mathrm{Ass}(\mathrm{Hom}_A(M, N))$.

14. Let $\rho : A \to B$ be a ring homomorphism, M an A-module, and N a B-module which is flat when regarded as an A-module. Prove that

$$\mathrm{Ass}_B(M \otimes_A N) \supset \bigcup_{P \in \mathrm{Ass}_A(M)} \mathrm{Ass}_B(N/PN),$$

with equality if A is Noetherian. What can you say if you replace Ass with $\mathrm{Ass}^{\mathrm{AM}}$?

References

[1] Atiyah, M.F. and I.G. MacDonald, *Introduction to Commutative Algebra*. New York: Addison-Wesley Publising Company, 1969.

[2] Bourbaki, Nicolas, *Commutative Algebra*. Paris: Hermann, Publishers in Arts and Science, 1972.

[3] Matsumura, Hideyuki, *Commutative Ring Theory*. Cambridge: Cambridge University Press, 1980.