1. *Short answer questions; 2 points each*

(a) Find the 101st term of the arithmetic sequence whose 2nd term is 7 and 5th term is -5.

Common difference is -4

$\Rightarrow$ first term is 11

The n^{th} term is given by $11+(-4)(n-1)$

$\Rightarrow 101^{st}$ term is $11+(-4)\cdot 100 = \boxed{-389}$

(b) Negate the statement: "Some exams are not enjoyable."

All exams are enjoyable.

or

No exams are not enjoyable.

(c) Give the contrapositive of the statement: "If I get a perfect score on this exam, then I will treat the professor to ice cream."

(Note: I accepted variants on the tense of the sentence, e.g. "If I do not treat the professor to ice cream, then I will not get a perfect score on this exam.")

If I will not treat the professor to ice cream, then I will not have gotten a perfect score on this exam.

(d) Give a numerical example of the associative property of multiplication.

e.g. $(2\cdot 3)\cdot 4 = 2\cdot(3\cdot 4)$

(e) State the distributive property of multiplication over addition.

If a, b, c are integers, then

$a\cdot(b+c) = a\cdot b + a\cdot c$

(f) If A is the set of books by authors whose last name is Austen, and if J is the set of books by authors whose first name is Jane, what is $A \cap J$?

$A \cap J$ = the set of books by authors whose first name is Jane and whose last name is Austen

(or: the set of books by Jane Austen)

(g) Let f be a function on the set of pairs of whole numbers, given by the rule $f(x, y) = x^2 + y^2$. Let g be a function on the set of whole numbers, given by the rule $g(z) = (1, 2z)$. Compute $(f \circ g)(4)$.

$$g(4) = (1, 2 \cdot 4) = (1, 8)$$

$$\Rightarrow (f \circ g)(4) = f(g(4)) = f(1, 8)$$
$$= 1^2 + 8^2 = \boxed{65}$$

(h) What number precedes 3400_{six} in base six?

$\boxed{3355_{six}}$

$$\begin{array}{r} 3400 \\ -\ 1 \\ \hline 3355 \end{array}$$

(i) Factor the expression $x^2 + 2x + 1 - 25(y^2)$.

$$\underbrace{x^2 + 2x + 1}_{(x+1)^2} - 25(y^2) = (x+1)^2 - (5y)^2$$
$$= \boxed{(x+1+5y)(x+1-5y)}$$

(j) Write the fraction $\frac{1}{3} + \frac{1}{5} - \frac{1}{30}$ in lowest terms.

$$\frac{1}{3} + \frac{1}{5} - \frac{1}{30} = \frac{10}{30} + \frac{6}{30} - \frac{1}{30}$$
$$= \frac{15}{30} = \boxed{\frac{1}{2}}$$

in lowest terms

(k) Write the fraction $\frac{17}{20} \div \frac{4}{5}$ in lowest terms.

$$\frac{17}{20} \div \frac{4}{5} = \frac{17}{20} \times \frac{5}{4} = \frac{17 \times 5}{80} = \frac{17 \times 5}{16 \times 5} = \boxed{\frac{17}{16}}$$

(l) If I flip a fair coin three times, what is the probability that exactly two of the flips are heads?

Oops! I starred the possibilities with two tails, instead of those with two tails. It should be HHT, HTH, and THH that are starred.

The possibilities are:

HHH	THH
HHT	THT*
HTH	TTH*
HTT*	TTT

$\Rightarrow$ Probability is $\boxed{3/8}$

(m) Compute the mean of the three numbers: 1, 4, 10.

$$\text{Mean} = \frac{1+4+10}{3} = \frac{15}{3} = \boxed{5}$$

(n) Compute the variance of the three numbers: 1, 4, 10.

Alternately, you can compute the average of the squares (1^2 + 4^2 + 10^2)/3 = 117 = 39, so the variance is 39 - 5^2 = 14.

$$\text{Variance} = \frac{(1-5)^2 + (4-5)^2 + (10-5)^2}{3}$$

$$= \frac{16+1+25}{3} = \frac{42}{3} = \boxed{14}$$

(o) Compute the standard deviation of the three numbers: 1, 4, 10.

Std Dev = square root of variance

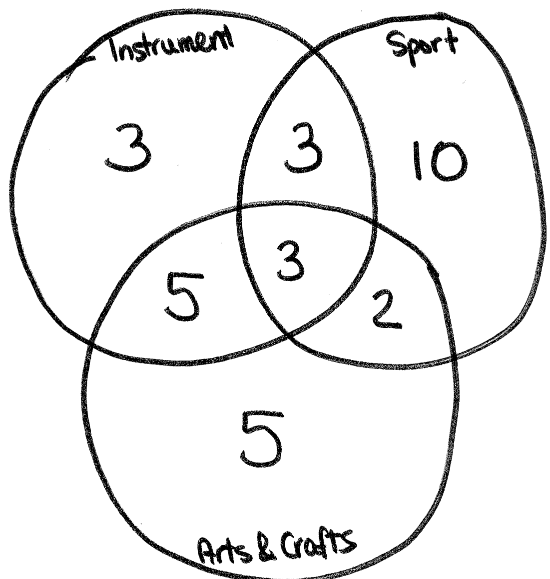
$$= \boxed{\sqrt{14}}$$

2. In a certain grade 6 class, every student either plays an instrument, plays a sport, or enjoys arts and crafts. Suppose that

   - 14 students play an instrument,
   - 18 students play a sport,
   - 15 students enjoy arts and crafts,
   - 6 students both play an instrument and play a sport,
   - 8 students both play an instrument and enjoy arts and crafts,
   - 5 students both play a sport and enjoy arts and crafts,
   - 3 students do all three.

   So, for example, there are exactly two students who play a sport, enjoy arts and crafts, and do not play an instrument. What is the total number of students in the class?

Venn Diagram:



$3+3+10+5+3+2+5 = \boxed{31 \text{ total students}}$

or

$14+18+15-6-8-5+3 = 31$ total students

3. Dr. Wonderful gave a seminar at McGill last semester. Before her talk, she asked a volunteer to participate in the following trick:

Step 1: Pick a whole number.
Step 2: Multiply this number by 3.
Step 3: Add 1 to the result of Step 2.
Step 4: Add 1 to the result of Step 3.
Step 5: Multiply the result of Step 3 by the result of Step 4.
Step 6: Calculate the remainder when the result of Step 5 is divided by 18.

She then amazed the volunteer by predicting the result of Step 6! What was Dr. Wonderful's prediction? Explain why the trick works.

The prediction is $\boxed{2}$.

Here's how the trick works.

Step 1: n
Step 2: $3n$
Step 3: $3n+1$
Step 4: $3n+2$
Step 5: $(3n+1)(3n+2) = 9n^2+9n+2$.

Now, in Step 6, we observe that $9n^2+9n$ is always a multiple of 18: it is obviously a multiple of 9, but it is also even, because n^2 and n are either both even or both odd.

Since $9n^2+9n$ is a multiple of 18, the remainder when $9n^2+9n+2$ is divided by 18 is $\boxed{2}$.

Having the expression 9n^2 + 9n + 2 somewhere on the page, and stating that the remainder was 2, was worth 8 points out of 10.

4. (a) (*3 points*) If a, m, and n are natural numbers, explain why $a^m \cdot a^n = a^{m+n}$.

$$a^m = \underbrace{a \cdot a \cdot \dots \cdot a}_{m \text{ times}}$$

$$a^n = \underbrace{a \cdot a \cdot \dots \cdot a}_{n \text{ times}}$$

$$\Rightarrow a^m \cdot a^n = \underbrace{a \cdots a}_{m \text{ times}} \cdot \underbrace{a \cdots a}_{n \text{ times}} = \underbrace{a \cdots a}_{m+n \text{ times}} = a^{m+n}.$$

(b) (*4 points*) Give a reason why, if a is a natural number, we choose to define $a^0 = 1$.

Sol$^{\underline{n}}$ 1: As explained in class, we get from a^m to a^{m+1} by multiplying by a. So we should get from a^0 to $a^1 = a$ by multiplying by $a \Rightarrow a^0$ should be 1.

Sol$^{\underline{n}}$ 2: If we define $a^0 = 1$, then the formula in part (a), which we have explained for $m, n \geq 1$, is also valid if m or n is zero.

(c) (*2 points*) Write $3^{20} \cdot 3^{20}$ in the form a^b in *two different ways.*

1) $3^{20} \cdot 3^{20} = 3^{40}$ (same bases $\Rightarrow$ exponents add)

2) $3^{20} \cdot 3^{20} = 9^{20}$ (same exponents $\Rightarrow$ bases multiply)

3) $3^{20} \cdot 3^{20} = (3^{20})^2$

5. (*2 points each*)

(a) Compute the product $11_{seven} \times 1234_{seven}$.

$$\begin{array}{r} 1234_{seven} \\ 11_{seven} \\ \hline 1234 \\ 1234 \\ \hline 13604_{seven} \end{array} \Rightarrow \boxed{13604_{seven}}$$

(b) What is the remainder when 100_{seven} is divided by 6?

$$\begin{array}{r} 11 \\ 6\overline{)100} \\ \underline{6} \\ 10 \\ \underline{6} \\ 1 \end{array}$$

or

$$100_{seven} = 66_{seven} + 1$$
$$= 6 \cdot 11_{seven} + 1$$

$\Rightarrow$ remainder $= \boxed{1}$

(c) What is the remainder when 1000_{seven} is divided by 6?

$$\begin{array}{r} 111 \\ 6\overline{)1000} \\ \underline{6} \\ 10 \\ \underline{6} \\ 10 \\ \underline{6} \\ 1 \end{array}$$

or

$$1000_{seven} = 666_{seven} + 1$$
$$= 6 \cdot 111_{seven} + 1$$

$\Rightarrow$ remainder $= \boxed{1}$

(d) Given a number in base seven, can you suggest a test for divisibility by 6?

Add up the digits of the number in base seven and see if this is divisible by 6.

(e) Given a number in base eight, can you suggest a test for divisibility by 7?

Add up the digits of the number in base eight and see if this is divisible by 7.

6. (*2 points each*)

(a) If a does not divide b and a does not divide c, then a cannot divide $b+c$. True or false? If true, explain why; if false, provide a counterexample.

False.

e.g. $2 \nmid 3$, $2 \nmid 5$ but $2 \mid 8 (= 3+5)$

$(a=2, b=3, c=5)$

(b) If a does not divide b and b does not divide c, then a cannot divide c. True or false? If true, explain why; if false, provide a counterexample.

False.

e.g. $2 \nmid 3$, $3 \nmid 4$ but $2 \mid 4$.

$(a=2, b=3, c=4)$

(c) Write 96 and 120 as a product of primes.

$96 = 2^5 \cdot 3^1$

$120 = 2^3 \cdot 3^1 \cdot 5^1$

(d) Compute GCD(96, 120).

$\text{GCD}(96,120) = 2^3 \cdot 3^1 \cdot 5^0$

$= \boxed{24}$

(e) Compute LCM(96, 120).

$\text{LCM}(96,120) = 2^5 \cdot 3^1 \cdot 5^1 = \boxed{480}$

(or: $\text{LCM} = \frac{96 \cdot 120}{24} = 480$)

7. (*5 points for each part*) Read the newspaper article on the following page. You may tear the next page out of this booklet, if it is more convenient for you. (Also, you may stop reading at the paragraph which begins "OBVIOUSLY".)

(a) Justify Professor Fox's conclusion that contestants with numbers in the range 00010 through 09999 had a one in 20,000 chance of being the winner.

We make a branch of the probability tree for a number in this range.

1st digit is correct	2nd dig. is correct	3rd dig. correct	4th dig. correct	5th dig. correct
$1/2$	$1/10$	$1/10$	$1/10$	$1/10$

0, 1 are the poss. for the 1st digit

if 0 is the 1st digit, next digit is anything from 0 to 9

all may be 0–9, just like the 2nd digit.

$$\Rightarrow \frac{1}{2}\cdot\frac{1}{10}\cdot\frac{1}{10}\cdot\frac{1}{10}\cdot\frac{1}{10} = \boxed{\frac{1}{20000}}.$$

(b) Justify Professor Fox's conclusion that contestants with numbers in the range 11500 through 11559 had a one in 1,680 chance of being the winner.

For a number in this range,

chance that first digit is correct $= \frac{1}{2}$ (possibilities are 0 or 1)

if 1st digit is 1, chance that 2nd dg. correct $= \frac{1}{2}$ (possibilities: 0 or 1)

if 1st two digits are 11, chance that 3rd digit is 5 $= \frac{1}{6}$ (3rd digit is chosen from 0..5)

if first three digits are 115, chance 4th digit is correct is $1/7$ (4th digit may be between 0 and 6)

chance that last digit is correct is $1/10$ (5th digit is chosen from 0...9)

$$\Rightarrow \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{6}\cdot\frac{1}{7}\cdot\frac{1}{10} = \boxed{\frac{1}{1680}}$$

NATIONAL NEWS

Prof says lottery drawing not fair

By KEITH GAVE
Staff Writer

Not everyone has an equal chance of becoming a millionaire because the state lottery bureau uses an unfair procedure to select finalists for its million-dollar drawing, a Michigan State University professor of statistics and probability charges.

And if the system is not changed, a select few contestants could have an overwhelming advantage over the majority in a similar drawing among qualifiers—$50 winners—in the current instant lottery game, "Match Two," MSU's Martin Fox said.

FOX STUDIED the probability of certain five-digit numbers being selected from a pool of 11,561 combinations—which represent the number of possible contestants in the Dec. 30, 1981 drawing—which determined the five finalists for Thursday's million-dollar drawing in Flint's Hyatt Regency.

Fox said his analysis, based on procedures set by a Bureau of State Lottery directive concluded that not every $50 instant game winner who qualifies for the million-dollar drawing has the same chance of winning.

In the game "Aces Three" that began Sept. 9, there were 11,561 instant $50 winning tickets that could have qualified for the December elimination drawing to determine the finalists and several alternates.

BUREAU RULES require each $50 qualifier to file a form with the agency in order to collect a check for that amount. Accompanying the $50 check is a five-digit number assigned in chronological order to each contestant. In the Aces Three contest these numbers range from 00001 to 11,561. These are the numbers the qualifiers hold in the drawing which determine the five finalists.

According to Fox's findings:

—Those contestants assigned the five digit numbers 00001 through 00009 had a one in 18,000 probability of being selected as one of the five finalists.

—Those with numbers 00010 through 09999 had a one in 20,000 probability of being selected.

—Those with numbers 10000 through 10999 had a one in 4,000 probability of being selected.

—Those with numbers 11000 through 11499 had a one in 2,400 probability of being selected.

—Those with numbers 11500 through 11559 had a one in 1,680 probability of being selected.

—Contestants who were assigned the two remaining numbers, 11560 and 11561, had a one in 336 probability. "That's pretty good," Fox noted.

FOX, WHO has been with MSU for 22 years, took issue with the bureau's system of drawing numbers from a pool and returning to the pool those numbers that fall out of the range needed to complete the five-digit number.

In the Dec. 30 drawing, for example, the first two digits of one number were 11. If the third number drawn was a 6, 7, 8 or 9, it was considered "out of range," tossed back into the machine and redrawn. The process continued until a five-digit number fell within the range of 00001 to 11,561.

Fox argued that for absolute fairness, if one digit of the number is tossed back into the machine, then all digits of the number drawn so far should be tossed back. He admitted that getting five five-digit numbers using that system might be a lengthy and tedious process. Nevertheless, he said, before the first number even pops out of the bingo blower machine, each contestant would have an equal chance of becoming a finalist.

OBVIOUSLY, according to his conclusions, those who bought their $50 ticket and collected their winnings—and their number—earlier in the contest had a lesser chance of winning than those who were among the last to file.

Lottery officials noted, however, that only 11,067 of the 11,561 $50 instant tickets were sold. And of those, only 10,256 were claimed by contestants who were then mailed a five-digit number.

Fox said figuring the probabilities for those numbers would be "very complicated," without knowing exactly which numbers were missing from the range.

"But my guess, and it's only a guess, is that the gaps are roughly uniformly spread," he said, "which would preserve roughly the same unfairness."

FOX SAID his calculations were based on a bureau directive he called "well written and easier to understand than most government documents.

"If they mean what they're saying, then my opinion is correct and the procedure is very unfair to people (who are assigned) low numbers," he said.

Fox made his analysis at the request of Erwin R. Braker, one of the contestants in last month's elimination drawing and one of a handful to attend the ceremony to select the finalists.

After the first two digits were drawn (for each of the five finalists), Braker was eliminated. He said he left with an unsettling feeling that the process of selecting the numbers was biased. His number was 00400. He later inquired about the process to a lottery bureau official, who referred him to experts at MSU.

8. Here list a list of the final exam scores from McGill's Organic Chemistry class: (the Chemistry department gives hard exams!)

20 25 45 45 50 50 52 54 55 58 60 62 64 65 65 65 68 74 77 85 96 98

(a) (*2 points*) Calculate the median and mode of this data.

$$\text{median} = \frac{60+62}{2} = \boxed{61}$$

$$\text{mode} = \boxed{65}$$

(b) (*3 points*) Calculate Q_1, Q_3, and the IQR for this data.

$$Q_1 = \boxed{50}$$

$$Q_3 = \boxed{68}$$

$$IQR = Q_3 - Q_1 = \boxed{18}.$$

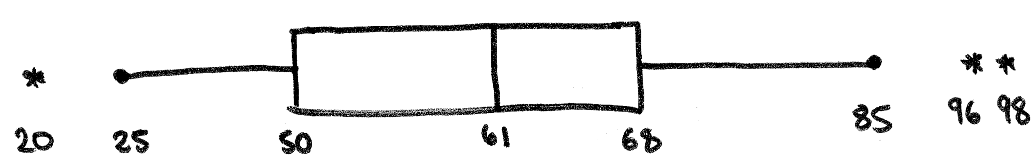
(c) (*2 points*) Identify which of the data points (if any) are outliers.

outliers lie above $Q_3 + 1.5 \times IQR = 68 + 1.5 \times 18 = 95$

or below $Q_1 - 1.5 \times IQR = 50 - 1.5 \times 18 = 23$

$\Rightarrow \boxed{20, 96, 98}$ are outliers.

(d) (*3 points*) Draw a box-and-whisker plot of this data.



Bonus #1 (*Up to 5 bonus points*)

At the beginning of April, the United States Department of Labor released its monthly employment report, which indicated that the number of jobs in the United States increased by 308,000 in March. In response, President Bush was quoted as saying: "This week we received powerful confirmation that America's economy is growing stronger." List as many items of additional information as you can think of, that would be useful to you in evaluating whether Bush's response to this statistic is justified.

Some possibilities:

* The population of the U.S. is increasing. What level of job growth is needed just to keep up with population growth?
* Is one month of data significant? If Jan, Feb numbers were lower, maybe this is normal fluctuation and does not represent a growing economy.
* What sort of jobs were created? Some jobs are seasonal; what sort of job growth is typical each March?
* What level of job creation indicates a stronger economy? Surely not 3 jobs, or 3000 jobs... and certainly 3000000 jobs... are 308000 jobs significant relative to the size of the U.S.?
* Were the new jobs created by gov't legislation?

Sorry, not going to give you the answer to this one! You should try it yourselves!

I have had some trouble getting this page to print properly -- I hope it works for you.

Bonus #2 (*Up to 5 bonus points*)

There is a complete set of 28 dominos hidden in the plus-shaped diagram below. The edges of each domino have been erased. Draw in the edges to indicate exactly where each domino lies. A list of the dominos to be found is given below; note, however, that the dominos may be rotated. An example (including solution) with 10 dominos is given.

Example:

2	2	1	2	3
1	0	3	3	0
2	3	0	1	1
0	1	0	3	2

2	2	1	2	3
1	0	3	3	0
2	3	0	1	1
0	1	0	3	2

0-0
0-1 1-1
0-2 1-2 2-2
0-3 1-3 2-3 3-3

Your puzzle:

	6	5	2	2	6	3	
	6	1	0	6	1	3	
1	1	1	0	4	5	3	1
4	6	6	0	6	6	3	1
4	4	2	0	2	4	3	4
2	2	2	0	1	4	3	2
	5	5	0	3	0	3	
	5	5	0	4	5	5	

0-0
0-1 1-1
0-2 1-2 2-2
0-3 1-3 2-3 3-3
0-4 1-4 2-4 3-4 4-4
0-5 1-5 2-5 3-5 4-5 5-5
0-6 1-6 2-6 3-6 4-6 5-6 6-6