MATH 111 SOLUTIONS FOR HW 9

1. Write each of the following numbers as a product of primes:
   - $28 = 2 \times 2 \times 7 = 2^2 \cdot 7$
   - $48 = 2 \times 2 \times 2 \times 2 \times 3 = 2^4 \cdot 3$
   - $60 = 2 \times 2 \times 3 \times 5 = 2^2 \cdot 3 \cdot 5$
   - $91 = 7 \times 13$
   - $121 = 11 \times 11 = 11^2$
   - $1331 = 11 \times 11 \times 11 = 11^3$
   - $10201 = 101 \times 101 = 101^2$
   - $1030301 = 101 \times 101 \times 101 = 101^3$

2. Find the following greatest common divisors:
   - $\mathrm{GCD}(28, 60) = 2^2 = 4$
   - $\mathrm{GCD}(28, 91) = 7$
   - $\mathrm{GCD}(48, 91) = 1$
   - $\mathrm{GCD}(121, 1331) = 11^2 = 121$

3. Find the following least common multiples:
   - $\mathrm{LCM}(28, 60) = 2^2 \cdot 3 \cdot 5 \cdot 7 = 420$
   - $\mathrm{LCM}(28, 91) = 2^2 \cdot 7 \cdot 13 = 364$
   - $\mathrm{LCM}(48, 91) = 2^4 \cdot 3 \cdot 7 \cdot 13 = 4368$
   - $\mathrm{LCM}(121, 1331) = 11^3 = 1331$

4. For each of the pairs $(28, 60)$, $(28, 91)$, $(48, 91)$, and $(121, 1331)$, verify the equation $\mathrm{GCD}(a, b) \times \mathrm{LCM}(a, b) = a \times b$.
   - $\mathrm{GCD}(28, 60) \times \mathrm{LCM}(28, 60) = 4 \times 420 = 1680 = 28 \times 60$
   - $\mathrm{GCD}(28, 91) \times \mathrm{LCM}(28, 91) = 7 \times 364 = 2548 = 28 \times 91$
   - $\mathrm{GCD}(48, 91) \times \mathrm{LCM}(48, 91) = 1 \times 4368 = 4368 = 48 \times 91$
   - $\mathrm{GCD}(121, 1331) \times \mathrm{LCM}(121, 1331) = 121 \times 1331$

5. Can you explain why the equation $\mathrm{GCD}(a, b) \times \mathrm{LCM}(a, b) = a \times b$ is always true?

   Many of you wrote something like "Because every time we try it, it works"; but this avoids the question. The thing we want to know is precisely, why does it work every time?

   Others wrote that because

   $$\mathrm{LCM}(a, b) = \frac{a \times b}{\mathrm{GCD}(a, b)},$$

   the equation $\mathrm{GCD}(a, b) \times \mathrm{LCM}(a, b) = a \times b$ follows. However, this is circular reasoning: if you look at your notes from class, you will see that we concluded $\mathrm{LCM}(a, b) = \frac{a \times b}{\mathrm{GCD}(a,b)}$ from the equation $\mathrm{GCD}(a, b) \times \mathrm{LCM}(a, b) = a \times b$ in the first place. So the question still remains: why is the equation $\mathrm{GCD}(a, b) \times \mathrm{LCM}(a, b) = a \times b$ true?

Let us look at a specific example first: $\text{GCD}(28, 91) \times \text{LCM}(28, 91) = 7 \times (2^2 \cdot 7 \cdot 91)$, while $28 \times 91 = (2^2 \cdot 7) \times (7 \cdot 13)$. The same primes occur the same number of times each, in these products. For example, 2 occurs twice in the prime factorization of 28 and no times in 91; while 2 occurs no times in the GCD and twice in the LCM. The prime 7 occurs once each in 28 and 91, and it also occurs once each in the GCD and LCM.

Let's think about a slightly more abstract example. Suppose there's a prime that occurs exactly five times in the prime factorization of a, and that this same prime occurs exactly three times in the prime factorization of b. (For example, a might be $96 = 2^5 \cdot 3$, and b might be $120 = 2^3 \cdot 3 \cdot 5$. Then when $a \times b$ is written as a product of primes, the number of times that this prime occurs is $5 + 3 = 8$. (For example, $96 \cdot 120 = 2^8 \cdot 3^2 \cdot 5$.)

But when we calculate the GCD of a and b, we include this prime exactly three times (three being the smaller of three and five); for example, $\text{GCD}(96, 120) = 2^3 \cdot 3 = 24$. Similarly, when we calculate the LCM, we include this prime exactly five times; for example, $\text{LCM}(96, 120) = 2^5 \cdot 3 \cdot 5 = 480$. So this prime also occurs 8 times in the product of the GCD and the LCM. In this manner, every prime occurs the same number of times in $a \cdot b$ as it does in $\text{GCD}(a, b) \cdot \text{LCM}(a, b)$, and so these two numbers are equal.

Here's a more mathematically complete solution, along these lines: suppose the prime p occurs k times in the prime factorization of a, and l times in the prime factorization of b. Then p occurs $k + l$ times in the prime factorization of $a \cdot b$. Also, the prime occurs in the prime factorization of $\text{GCD}(a, b)$ a number of times equal to the minimum of k and l, and it occurs in the factorization of $\text{LCM}(a, b)$ a number of times equal to the maximum of k and l. So p occurs in $\min(k, l) + \max(k, l)$ times in the prime factorization of $\text{GCD}(a, b) \times \text{LCM}(a, b)$. But $\min(k, l) + \max(k, l) = k + l$, so p occurs in the prime factorizations of $a \cdot b$ and $\text{GCD}(a, b) \times \text{LCM}(a, b)$ the same number of times each. Therefore the prime factorizations of $a \cdot b$ and $\text{GCD}(a, b) \times \text{LCM}(a, b)$ are the same, and so these two numbers are equal.