# MATH 215: CHAPTER 4 HOMEWORK SOLUTIONS

**4.1 #2.**

(a) The set of all column vectors in $\mathbb{R}^3$ of length 1 is **not** a vector space, for many reasons. A vector space must contain the zero vector, but the zero vector does not have length 1. The scalar product of $c \in \mathbb{R}$ and a vector of length 1 has length $|c|$, which is not 1 unless $c = \pm 1$. Finally, the sum of two vectors of length 1 does not necessarily have length one, e.g.

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

and the length of $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is $\sqrt{2}$.

(b) The set of polynomials of degree at least 2 is **not** a vector space. For instance, once again this set does not contain the zero polynomial (as the zero polynomial has degree $-\infty$). Alternatively, we can see that the sum of two polynomials of degree at least 2 does not necessarily have degree at least 2: e.g. $(x^2 + 1) + (-x^2 + x) = x + 1$, and $x + 1$ has degree 1.

(c) The sum of two vectors parallel to a given plane is again parallel to that plane. A scalar multiple of a vector parallel to a given plane is again parallel to it. The zero vector is considered parallel to any plane. So, the set of all line segments parallel to a given plane **is** a vector space.

Here is a way to formalize this. Let $\vec{n}$ be a vector which is normal to the given plane. Then $\vec{v}$ is parallel to the plane if and only if $\vec{v} \cdot \vec{n} = 0$. If $\vec{v} \cdot \vec{n} = 0$ and $\vec{w} \cdot \vec{n} = 0$ then $(\vec{v} + \vec{w}) \cdot \vec{n} = 0$; if $\vec{v} \cdot \vec{n} = 0$ then $(c\vec{v}) \cdot \vec{n} = 0$; and $0 \cdot \vec{n} = 0$.

(d) The set of all conditions functions of $x$ defined on the interval $[0, 1]$ such that $f(1/2) = 0$ **is** a vector space. This set certainly contains the zero function. Since the set of all continuous functions of $x$ defined on the interval $[0, 1]$ (with no assumption about $f(1/2)$) is a vector space, all we have to check is that if $f(1/2) = 0$ and $g(1/2) = 0$ then $(f + g)(1/2) = 0$ and $(cf)(1/2) = 0$, which is clear.

**4.2 #3.** We want to determine whether or not there exist scalars $c_1, c_2, c_3$ such that

$$1 - 2x + x^2 = c_1(1 + x^2) + c_2(x^2 - x) + c_3(3 - 2x)\,.$$

Collecting terms terms of the same degree gives

$$1 - 2x + x^2 = (c_1 + 3c_3) + (-c_2 - 2c_3)x + (c_1 + c_2)x^2$$

so we want to know whether the system of equations

$$\begin{aligned} c_1 + 3c_3 &= 1 \\ -c_2 - 2c_3 &= -2 \\ c_1 + c_2 &= 1 \end{aligned}$$

has a solution. Using our usual methods of solving systems of linear equations, we find that there is a solution: $c_1 = -1/5$, $c_2 = 6/5$, $c_3 = 2/5$. So

$$1 - 2x + x^2 = \frac{-1}{5}(1 + x^2) + \frac{6}{5}(x^2 - x) + \frac{2}{5}(3 - 2x)\,,$$

and $1 - 2x + x^2$ is in the subspace generated by $1 + x^2,\ x^2 - x,\ 3 - 2x$.

**4.2 #4**. We want to determine whether or not there exist scalars $c_1, c_2, c_3$ such that

$$\begin{pmatrix} 4 & 3 \\ 1 & -2 \end{pmatrix} = c_1 \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 2 \\ -1/3 & 4 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 2 \\ 6 & 1 \end{pmatrix}.$$

Looking at each entry of the matrix, we want to determine whether the system of equations

$$\begin{aligned} 3c_1 &= 4 \\ 4c_1 + 2c_2 + 2c_3 &= 3 \\ c_1 - c_2/3 + 6c_3 &= 1 \\ 2c_1 + 4c_2 + c_3 &= -2 \end{aligned}$$

has a solution. By our usual methods of solving systems of linear equations, we find that this system is inconsistent, and so the matrix is not in the given subspace.

**4.2 #7**. If $A$ is a scalar multiple of $B$ or vice-versa, then $A$ and $B$ certainly do not generate all of $\mathbb{R}^2$: this is because $A$ and $B$ lie on the same line, and every linear combination of $A$ and $B$ lies on that line as well.

Now we have to show that if $A$ and $B$ are not scalar multiples of one another, then they do generate all of $\mathbb{R}^2$. Here is one way to do it. Let $A = (s, t)$ and $B = (u, v)$. First, prove that $A$ and $B$ are scalar multiples of one another if and only if $sv - tu = 0$. (Check this!!) So we know $sv - tu \neq 0$; that is, the determinant of

$$M = \begin{pmatrix} s & t \\ u & v \end{pmatrix}$$

is not zero, and so $M$ is invertible. We want to show that $A, B$ generate $\mathbb{R}^2$; that is, we want to show that the system of equations

$$c_1 A + c_2 B = (x, y)$$

in the variables $c_1, c_2$ has a solution for *any* $(x, y)$. This is the system of equations

$$\begin{aligned} c_1 s + c_2 t &= x \\ c_1 u + c_2 v &= y\,, \end{aligned}$$

i.e. it is the system of equations

$$M \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Since $M$ is invertible, the system does have a (unique) solution.

**4.3 #1**.

(a) We want to decide whether or not there exist $c_1, c_2, c_3$, not all zero, such that

$$c_1 \begin{pmatrix} -1 \\ 2 \\ \sqrt{-1} \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 3+\sqrt{-1} \\ -2 \end{pmatrix} + c_3 \begin{pmatrix} -5 \\ 3\sqrt{-1} \\ -6+2\sqrt{-1} \end{pmatrix} = 0\,.$$

That is, we want to know whether the system of equations

$$\begin{pmatrix} -1 & 1 & -5 \\ 2 & 3+\sqrt{-1} & 3\sqrt{-1} \\ \sqrt{-1} & -2 & -6+2\sqrt{-1} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = 0$$

has a nontrivial solution. By Theorem 2.3.5 on page 59, this is the case if and only if the matrix

$$\begin{pmatrix} -1 & 1 & -5 \\ 2 & 3+\sqrt{-1} & 3\sqrt{-1} \\ \sqrt{-1} & -2 & -6+2\sqrt{-1} \end{pmatrix}$$

is not invertible. But the determinant of this matrix is $44+5*\sqrt{-1}$, so it is invertible, and the set $S$ is linearly independent.

(b) It is easy to check that

$$a(x^3 - x^2 - x + 3) + b(x^2 + 1) + c(x - 1) = 0$$

implies $a = b = c = 0$. (First $a = 0$ because the $x^3$ term must be zero, then $b = 0$ because of the $x^2$ term, then $c = 0$.) So the set $S$ is linearly independent.

(c) Since

$$\begin{pmatrix} 12 & -7 \\ 17 & 6 \end{pmatrix} = 3 \begin{pmatrix} 2 & -3 \\ 6 & 4 \end{pmatrix} + 2 \begin{pmatrix} 3 & 1 \\ -1/2 & -3 \end{pmatrix}$$

the set $S$ is linearly dependent. But how did we find this? Use the method that we used in (a) and (b): turn the problem into a system of equations and solve the system, but this time we find that there is a nontrivial solution. We are considering the matrix

$$\begin{pmatrix} 2 & 3 & 12 \\ -3 & 1 & -7 \\ 6 & -1/2 & 17 \\ 4 & -3 & 6 \end{pmatrix}$$

and you can check that in row echelon form this matrix has 2 pivots. Since there are fewer pivots than columns, the set $S$ is dependent.

**4.3 #2**. True. If $X$ is a set containing 0, then

$$c \cdot 0 = 0$$

a linear dependence in $X$ for any nonzero $c$. (That is, multiply the zero vector by $c$, and every other vector in $X$ by zero.)

**4.3 #3**. True. A set $X$ is linearly independent if and only if every nontrivial linear combination of vectors in $X$ is nonzero. If $Y$ is a subset of $X$, then every nontrivial linear combination of vectors in $Y$ is also a nontrivial linear combination of vectors in $X$, so is not zero. Therefore $Y$ is linearly independent.

**4.3 #4**. False. For instance, suppose $X$ is the set in $\mathbb{R}^2$ containing $(1, 0)$, $(0, 1)$, and $(1, 1)$. This set is linearly independent. But the subset consisting of the two vectors $(1, 0)$ and $(0, 1)$ is linearly independent.

**4.3 #5, Solution 1**. Let $u, v, w$ be three vectors in $\mathbb{R}^2$. By 4.2 #7, either $u$ and $v$ are linearly dependent (in which case $u, v, w$ are certainly linearly dependent), or else $u$ and $v$ generate $\mathbb{R}^2$. But if $u, v$ generate $\mathbb{R}^2$, then there exist $c_1, c_2$ such that $c_1 u + c_2 v = w$, so

$$c_1 u + c_2 v - w = 0$$

which shows that $u, v, w$ are linearly dependent as well. So either way, these three vectors are linearly dependent.

**4.3 #5, Solution 2**. Let $u = (x_1, y_1)$, $v = (x_2, y_2)$, $w = (x_3, y_3)$ be three vectors in $\mathbb{R}^2$. We need to show that there exist $c_1, c_2, c_3$ not all zero such that

$$c_1 u + c_2 v + c_3 w = 0\,.$$

This is equivalent to the system of equations

$$\begin{aligned} x_1 c_1 + x_2 c_2 + x_3 c_3 &= 0 \\ y_1 c_1 + y_2 c_2 + y_3 c_3 &= 0\,. \end{aligned}$$

(Note that in this system of equations, the variables are $c_1, c_2, c_3$!) Since this is a homogeneous system of 2 equations in 3 unknowns, there is always a nontrivial solution, by Corollary 2.1.4 on p. 45 of the book.

The generalization is that $m$ vectors in $\mathbb{R}^n$ are always linearly dependent if $m > n$.

**4.3 #7**. Let $E_{ij}$ be the $m$-by-$n$ matrix which has a 1 in the $i, j$ entry and zeros elsewhere. Since there are $m$ choices for $i$ and $n$ choices for $j$, there are $mn$ such matrices. I claim that they are linearly independent. We have to prove that if

$$\sum_{i,j} a_{ij} E_{ij} = 0$$

then $a_{ij} = 0$ for all $i, j$. But this is clear, since $\sum_{i,j} a_{ij} E_{ij}$ is just the matrix $(a_{ij})$.

**4.3 #10**. **No!** For instance, suppose $u = (1, 0)$, $v = (0, 1)$, and $w = (1, 1)$ in $\mathbb{R}^2$. It is easy to see that set $\{u, v\}$ is linearly independent, as is $\{u, w\}$ and $\{v, w\}$. But $\{u, v, w\}$ is linearly *de*pendent, since $u+v-w=0$ is a linear dependence among them (or, because they are three vectors in $\mathbb{R}^2$, so are automatically dependent by 4.3 #5).