MATH 215: SECTION 5.1 HOMEWORK SOLUTIONS

#1.

(a) To show that three vectors form a bases of $\mathbb{R}^3$, it suffices to show that they are linearly independent. So, we need the matrix

$$A = \begin{pmatrix} 4 & -5 & 1 \\ 2 & 2 & 3 \\ 1 & -3 & 0 \end{pmatrix}$$

to have three pivots in row echelon form; equivalently, we need A to be invertible; equivalently, we need $\det(A) \neq 0$. It is easy to compute that $\det(A) = 13$, so the determinant is indeed not zero, and the vectors form a basis. The second part of the problem is asking us to solve the equations

$$A \begin{pmatrix} c_{11} \\ c_{21} \\ c_{31} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad A \begin{pmatrix} c_{12} \\ c_{22} \\ c_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad A \begin{pmatrix} c_{13} \\ c_{23} \\ c_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

You can solve these equations individually by the usual method, if you like. But here's another way: these three equations can be combined into the single equation

$$A \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and so

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} = A^{-1} = \frac{1}{13} \begin{pmatrix} 9 & -3 & -17 \\ 3 & -1 & -10 \\ -8 & 7 & 18 \end{pmatrix}$$

from which we get

$$E_1 = \frac{9}{13} X_1 + \frac{-3}{13} X_2 + \frac{-17}{13} X_3$$

and so forth.

(b) The method in the second part is exactly the same as in the first part. Let

$$B = \begin{pmatrix} 1 & 2 & 5 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{pmatrix}.$$

Then $\det(B) = -1$, so B is invertible, and is inverse is

$$B^{-1} = \begin{pmatrix} -2 & 4 & 1 \\ 4 & -7 & -3 \\ -1 & 2 & 1 \end{pmatrix}.$$

Therefore

$$E_1 = -2Y_1 + 4Y_2 - Y_3$$

and so forth.

#2.

(a) The nullspace of a matrix is the same as the nullspace of the reduced row echelon form of the matrix. The reduced row echelon form of the matrix in this problem is

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

There are two pivots, so the nullspace is one-dimensional. If

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

is a member of the nullspace, then $x_1 = -2x_3$ and $x_2 = -x_3$, so the nullspace is generated by

$$\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}.$$

(b) The reduced row echelon form of the given matrix is

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

There are two pivots and four columns, so the nullspace is two-dimensional. If

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

is a member of the nullspace, then $x_1 = x_3 - 2x_4$ and $x_2 = x_4 - x_3$, so a general element of the nullspace has the form

$$\begin{pmatrix} x_3 - 2x_4 \\ x_4 - x_3 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

These two vectors are a basis for the nullspace.

#3. The dimension of the vector space $M(m, n, \mathbb{R})$ is mn. To prove this, we will find a basis for it, and see that there are mn vectors in the basis. For each i, j with $1 \leq i \leq m$ and $1 \leq j \leq n$, let A_{ij} be the matrix all of whose entries are 0, except that its i, j entry is 1. There are mn of these matrices. We have already seen (4.3 #7) that these matrices are linearly independent. They also generate $M(m, n, \mathbb{R})$, because the matrix $A = (a_{ij})$ is equal to the linear combination

$$\sum_{i,j} a_{ij} A_{ij} \, .$$

Hence they are a basis.

#4. To prove that $v_1, \ldots, v_n$ form a basis of V, we have to prove that they generate V, and that they are linearly independent.

Show that $v_1, \ldots, v_n$ generate V: We need to show that every vector $v \in V$ is a linear combination of $v_1, \ldots, v_n$. But we are given this, and more! (We're given that every vector in V is a linearly combination of $v_1, \ldots, v_n$ in exactly one way.) So they certainly generate V.

Show that $v_1, \ldots, v_n$ are linearly independent: If

$$c_1 v_1 + \cdots + c_n v_n = 0$$

we must show that every $c_i = 0$. But every vector in V has only one way that it can be written as a linear combination of $v_1, \ldots, v_n$, and we already know that

$$0 \cdot v_1 + \cdots + 0 \cdot v_n = 0$$

is one way that 0 can be written as a linear combination of $v_1, \ldots, v_n$. So

$$c_1 v_1 + \cdots + c_n v_n = 0$$

must be the trivial linear combination, and so $c_1 = 0, \ldots, c_n = 0$.

#7. Let $v_1, \ldots, v_n$ be any basis of V; in particular they are linearly independent. Let

$$V_i = \langle v_1, \ldots, v_i \rangle \, ,$$

the span of the first i of the vectors. Then V_i is a subspace of V. But $v_1, \ldots, v_i$ are automatically linearly independent (e.g. by #3 from sec 4.3), so they are linearly independent and generate V_i, i.e. they are a basis of V_i. Since V_i has a basis of size i, it follows that $\dim(V_i) = i$.