MATH 215: SECTION 6.2 HOMEWORK SOLUTIONS

#1.

(a) T_1 is not a linear transformation. To show this, all we have to do is show that the rules for linear transformations are violated. For instance, $T_1([1\ 0\ 0]) = 1$ and $T_1([0\ 1\ 0]) = 1$, but

$$T_1([1\ 1\ 0]) = \sqrt{2} \neq 1 + 1 = T_1([1\ 0\ 0]) + T_1([0\ 1\ 0])\,.$$

So T_1 does not satisfy $T_1(v + w) = T_1(v) + T_1(w)$.

(b) T_2 is a linear transformation. Indeed, we know that $(A + B)^T = A^T + B^T$ (this is part (m) of Theorem 1.2.1 in the book) and $(cA)^T = cA^T$, which is what we need to check.

(c) T_3 is not a linear transformation, unless $n = 1$. We know that $\det(cA) = c^n \det(A)$, and when $n > 1$ this means that $\det(cA)$ is not always equal to $c \det(A)$. So T_3 is not linear when $n > 1$. When $n = 1$, T_3 is just the map $T_3([x]) = x$, which is obviously linear.

#2. We know that $-v = (-1)v$; this is Lemma 4.2.1(c) in the book. So

$$T(-v) = T((-1)v) = -1 \cdot T(v) = -T(v)\,.$$

#3. Here's one way to see this: if the line ℓ meets the x-axis at the angle ϕ, then reflection in the line ℓ is the composition of three linear transformations: rotate by $-\phi$, reflect in the x-axis, and then rotate back by ϕ. Since the composition of linear transformations is a linear transformation, we conclude that reflection in the line ℓ is also a linear transformation.

#4. We are looking for the matrix such that

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 - x_3 - x_4 \\ 2x_1 + x_2 - x_3 \\ x_2 - x_3 + x_4 \end{pmatrix}.$$

The matrix

$$\begin{pmatrix} 1 & -1 & -1 & -1 \\ 2 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix}$$

is obviously the one we want.

#5. We have

$$T(f+g) = x(f+g)'' - 2x(f+g)' + (f+g) = (xf'' - 2xf' + f) + (xg'' - 2xg' + g) = T(f) + T(g)$$

and similarly

$$T(cf) = x(cf)'' - 2x(cf) + (cf) = c(xf'' - 2xf + f) = cT(f)$$

so that T is a linear transformation. To find the matrix for T in terms of the basis $1, x, x^2, x^3$, we compute:

$$\begin{aligned} T(1) &= 1 \\ T(x) &= -x \\ T(x^2) &= 2x - 3x^2 \\ T(x^3) &= 6x^2 - 5x^3 \end{aligned}$$

If A is the matrix of the transformation, then for instance we must have $A \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 6 \\ -5 \end{pmatrix}$.

That is, the transformation matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -3 & 6 \\ 0 & 0 & 0 & -5 \end{pmatrix}.$$

#6, Solution 1. Writing points in $\mathbb{R}^2$ in polar coordinates, let's see what happens to the point (r, θ). The angle θ is $\phi + (\theta - \phi)$, and so the reflected point is $(r, \phi - (\theta - \phi)) = (r, 2\phi - \theta)$. In other words, the point $(x, y) = (r\cos(\theta), r\sin(\theta))$ is mapped to

$$\begin{aligned} &(r\cos(2\phi - \theta), r\sin(2\phi - \theta)) \\ &= (r\cos(\theta)\cos(2\phi) + r\sin(\theta)\sin(2\phi), r\cos(\theta)\sin(2\phi) - r\sin(\theta)\cos(2\phi)) \\ &= (x\cos(2\phi) + y\sin(2\phi), x\sin(2\phi) - y\cos(2\phi)) \end{aligned}$$

and so the matrix is

$$\begin{pmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{pmatrix}.$$

Notice that this matrix has determinant -1, as it should since it comes from a reflection.

#6, Solution 2. Use the fact that this transformation is rotation by $-\phi$, followed by reflection in the x-axis, followed by rotation by ϕ, so that the transformation matrix is the product

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

which is equal to

$$\begin{pmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{pmatrix}$$

(using the double-angle trig formulas).

#6, Solution 3. The transformation matrix maps $(1,0)$ to $(\cos 2\phi, \sin 2\phi)$, so the matrix must have the form

$$\begin{pmatrix} \cos(2\phi) & b \\ \sin(2\phi) & d \end{pmatrix}$$

for some b, d. The transformation also maps $(\cos 2\phi, \sin 2\phi)$ back to $(1,0)$, so

$$\cos^2(2\phi) + b\sin(2\phi) = 1$$

and

$$\sin(2\phi)\cos(2\phi) + d\sin(2\phi) = 0$$

so that $b = \sin(2\phi)$ and $d = -\cos(2\phi)$.

#7. First let's write down the change of basis from $\mathcal{B}'$ to the standard basis $\mathcal{B}$. We have

$$v_1 = 2e_1, \qquad v_2 = -e_1 + 2e_2, \qquad v_3 = e_1 + e_2 + e_3$$

and so the change of basis matrix is

$$\begin{pmatrix} 2 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

The change of basis from $\mathcal{B}$ to $\mathcal{B}'$ is simply the inverse of this, which is

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}.$$

#8. (We discussed this problem thoroughly in class.)

#9. Similar matrices have the same determinant; the first matrix has determinant 2 and the second has the determinant 10, so they cannot be similar.

#10. If B is similar to A, then there exists an invertible matrix X such that $B = XAX^{-1}$. Multiply on the left by X^{-1} and on the right by X, to see that $A = X^{-1}BX$. Writing this as

$$A = X^{-1}B(X^{-1})^{-1}$$

we see that A is similar to B.

#11. Since B is similar to A, there exists X such that $B = XAX^{-1}$. Since C is similar to B, there exists Y such that $C = YBY^{-1}$. Then

$$C = Y(XAX^{-1})Y^{-1} = (YX)A(X^{-1}Y^{-1}) = (YX)A(YX)^{-1},$$

so C is similar to A.

Extra question. We want to show that the map $f : \mathbb{R}^3 \to \mathbb{R}^2$ with $f(x,y,z) = (x,y)$ has a right-inverse but not a left-inverse.

First, let's show that it has a right-inverse: that is, we want a map $g : \mathbb{R}^2 \to \mathbb{R}^3$ such that $(f \circ g)(x, y) = (x, y)$. That's easy enough: just take $g(x, y) = (x, y, 0)$, which is a linear transformation, and

$$(f \circ g)(x, y) = f(g(x, y)) = f(x, y, 0) = (x, y)\,.$$

Next, let's show that there is no left-inverse: that is, there is no map $h : \mathbb{R}^2 \to \mathbb{R}^3$ such that $(h \circ f)(x, y, z) = (x, y, z)$. Suppose there were such a map. Then

$$(x, y, z) = (h \circ f)(x, y, z) = h(f(x, y, z)) = h(x, y)$$

and so we'd need to have $h(x, y) = (x, y, z)$ for all z! This is clearly impossible (we can't have $h(x, y)$ equal to both $(x, y, 0)$ and $(x, y, 1)$) so the left-inverse can't exist.

Another way to say this is, any function which has a left-inverse must be injective. Since f is not injective, it does not have a left-inverse.