

## MATH 215: SECTION 6.3 HOMEWORK SOLUTIONS

#1.

(a) This linear transformation is given by the matrix  $(1 \ 1 \ 1 \ 1)$ : this is because

$$(1 \ 1 \ 1 \ 1) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = (x_1 + x_2 + x_3 + x_4).$$

By the usual procedure, we can verify that a basis for the nullspace of  $(1 \ 1 \ 1 \ 1)$  is:

$$\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The image of this transformation is all of  $\mathbb{R}$ , so  $(1)$  is a basis.

(b) The kernel of this linear transformation is the set of constant polynomials, so, for instance, the polynomial 1 is a basis of the kernel.

The derivative of a polynomial of degree less than 3 is a polynomial of degree less than 2. Moreover,  $ax + b$  is the derivative of (for instance)  $\frac{1}{2}ax^2 + bx$ . So the image of the transformation is the set of all linear polynomials, and  $1, x$  is a basis.

(c)  $2x + 3y = 0$  if and only if  $4x + 6y = 0$  if and only if  $y = -\frac{2}{3}x$ , so the kernel is the set of vectors of the form  $\begin{pmatrix} x \\ -\frac{2}{3}x \end{pmatrix}$ . Hence  $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$  is a basis for the kernel.

Since  $(4x + 6y) = 2(2x + 3y)$ , every vector in the image has the form  $\begin{pmatrix} z \\ 2z \end{pmatrix}$ .

Therefore  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$  is a basis for the image.

#2. Let  $u_1, \dots, u_r$  be a basis for  $U$ . Since  $V$  is finite-dimensional, by Theorem 5.1.4 we can extend the basis of  $U$  to a basis of  $V$ : that is, we can find vectors  $u_{r+1}, \dots, u_n$  such that  $u_1, \dots, u_n$  is a basis of  $V$ .

Now define a linear transformation  $T : V \rightarrow V$  as follows. Each element  $v \in V$  can be written uniquely as a linear combination  $v = c_1u_1 + \dots + c_nu_n$ , and we define

$$T(v) = T(c_1u_1 + \dots + c_nu_n) = c_{r+1}u_{r+1} + \dots + c_nu_n.$$

We wish to prove that  $\ker(T) = U$ . First of all, if  $u \in U$  then since  $u_1, \dots, u_r$  are a basis for  $U$ ,  $u$  can be written as a linear combination of  $u_1, \dots, u_r$ ; therefore when we write  $u = c_1u_1 + \dots + c_nu_n$  we have  $c_{r+1}, \dots, c_n$  all equal to zero. So  $T(u) = 0 \dots u_{r+1} + \dots + 0 \dots u_n = 0$ , and  $u \in \ker(T)$ . Thus  $U \subset \ker(T)$ .

Conversely, suppose  $v \in \ker(T)$ , so that

$$T(v) = c_{r+1}u_{r+1} + \dots + c_nu_n0.$$

Since  $u_{r+1}, \dots, u_n$  are linearly independent, we must have  $c_{r+1}, \dots, c_n = 0$ . Therefore  $v = c_1u_1 + \dots + c_ru_r \in U$ , and so  $\ker(T) \subset U$ .

Since  $U \subset \ker(T)$  and  $\ker(T) \subset U$ , we find  $\ker(T) = U$ .

**#4.** We want to prove two things: (1) if  $T : V \rightarrow W$  is injective, then it maps linearly independent subsets of  $V$  to linearly independent subsets of  $W$ ; (2) if  $T : V \rightarrow W$  is not injective, then it does not map linearly independent subsets of  $V$  to linearly independent subsets of  $W$ .

Let's prove (2) first, since it's the easier of the two. If  $T : V \rightarrow W$  is not injective, we know that  $\ker(T) \neq 0$ , so there is a nonzero vector  $v \in V$  such that  $T(v) = 0$ . The set  $\{v\}$  is a linearly independent set; its image under  $T$  is  $\{T(v)\} = \{0\}$ , which is linearly dependent.

For (1), if  $v_1, \dots, v_r \in V$  are linearly independent, we have to show  $T(v_1), \dots, T(v_r)$  are linearly independent as well. Suppose

$$c_1T(v_1) + \dots + c_rT(v_r) = 0.$$

Then

$$T(c_1v_1 + \dots + c_rv_r) = 0$$

as well, and since  $T$  is assumed to be injective, we must have  $c_1v_1 + \dots + c_rv_r = 0$ . Since  $v_1, \dots, v_r$  are linearly independent, this shows  $c_1, \dots, c_r = 0$ . Hence  $T(v_1), \dots, T(v_r)$  are linearly independent, as desired.

**#5.** Let us begin by proving the following lemma:

**Lemma:** If  $f : V \rightarrow W$  is a linear transformation and  $V = \langle v_1, \dots, v_n \rangle$ , then  $\text{Im}(f) = \langle f(v_1), \dots, f(v_n) \rangle$ .

**Proof:** Since  $V = \langle v_1, \dots, v_n \rangle$ , we know that every element  $v \in V$  can be written as a linear combination  $v = c_1v_1 + \dots + c_nv_n$  for some real numbers  $c_1, \dots, c_n$ . But then

$$f(v) = c_1f(v_1) + \dots + c_nf(v_n).$$

Since  $w \in \text{Im}(f)$  if and only if  $w = f(v)$  for some  $v$ , it follows that  $w \in \text{Im}(f)$  if and only if  $w$  is a linear combination of  $f(v_1), \dots, f(v_n)$ . Therefore  $\text{Im}(f)$  is equal to the set of linear combinations of  $f(v_1), \dots, f(v_n)$ ; that is,  $\text{Im}(f) = \langle f(v_1), \dots, f(v_n) \rangle$ .

Now we can solve the problem at hand. Suppose that  $f : V \rightarrow W$  is surjective, and suppose that  $v_1, \dots, v_n$  generate  $V$ . We wish to prove that  $f(v_1), \dots, f(v_n)$  generate  $W$ . But this

follows immediately from the Lemma: indeed  $W = \text{Im}(f)$  since  $f$  is surjective, and so

$$W = \text{Im}(f) = \langle f(v_1), \dots, f(v_n) \rangle .$$

Conversely, suppose  $V = \langle v_1, \dots, v_n \rangle$  and  $W = \langle f(v_1), \dots, f(v_n) \rangle$ . We want to prove that  $f$  is surjective. But again using the Lemma,

$$\text{Im}(f) = \langle f(v_1), \dots, f(v_n) \rangle = W ,$$

so  $\text{Im}(f) = W$  and  $f$  is surjective.

**#9.** We are to prove that if  $T : V \rightarrow W$  and  $S : W \rightarrow U$  are isomorphisms, then  $ST : V \rightarrow U$  is also an isomorphism. But we know that the composition of two linear transformations is a linear transformation, so  $ST$  is a linear transformation; and the composition of two bijective functions is bijective, so  $ST$  is bijective. Therefore  $ST$  is a bijective linear transformation, i.e.,  $ST$  is an isomorphism.