MATH 215: CHAPTER 7 SOLUTIONS

7.2 #5. If $u = c_1v_1 + \cdots + c_nv_n$ and $w = d_1v_1 + \cdots + d_nv_n$, then in the notation of the problem,

$$[u] = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, \qquad [w] = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}.$$

Then

$$\begin{aligned} \langle u, w \rangle &= \langle c_1v_1 + \cdots + c_nv_n, d_1v_1 + \cdots + d_nv_n \rangle \\ &= \sum_{i=1}^{n} c_i \langle v_i, d_1v_1 + \cdots d_nv_n \rangle \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} c_id_j \langle v_i, v_j \rangle \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} c_id_ja_{ij} \qquad \text{since } a_{ij} = \langle v_i, v_j \rangle \\ &= [u]^T A [w], \end{aligned}$$

where the final equality follows from the definition of matrix multiplication.

7.2 #7. We have to prove that $\langle X, Y \rangle = X^TAY$ defines an inner product on $\mathbb{R}^n$. (The fact that $||X|| = (X^TAX)^{1/2}$ is a norm on $\mathbb{R}^n$ then follows automatically, since every inner product gives a norm.) Do to this we have to check several things.

- $\langle X, X \rangle \geq 0$: this is property (a) from #6.
- $\langle X, X \rangle = 0$ if and only if $X = 0$: this is property (b) from #6.
- $\langle X, Y \rangle = \langle Y, X \rangle$: since A is symmetric by (c), we have

  $$Y^TAX = Y^TA^TX = (X^TAY)^T = X^TAY$$

  where the last equality follows because X^TAY is just a number, or equivalently a 1-by-1 matrix, and every 1-by-1 matrix is automatically symmetric.
- $\langle X, cY \rangle = c\langle X, Y \rangle$: this follows from $X^TA(cY) = c(X^TAY)$.
- $\langle X, Y_1 + Y_2 \rangle = \langle X, Y_1 \rangle + \langle X, Y_2 \rangle$: this follows from $X^TA(Y_1 + Y_2) = X^TAY_1 + X^TAY_2$.

The problem not from the book. (i) Write $A = (a_{ij})$ and let v_j be the jth column of A; that is, v_j is the vector

$$\begin{pmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{pmatrix}.$$

By the definition of matrix multiplication, the ij-entry of $A^T A$ is

$$\sum_{k=1}^{n} a_{ki} a_{kj} = \langle v_i, v_j \rangle .$$

Thus $A^T A = I$ if and only if $\langle v_i, v_j \rangle$ is 1 if $i = j$ and 0 if $i \neq j$, i.e. if and only if the columns of A form an orthonormal set.

(ii) By definitoin, a matrix B such that $BA = I$ is the inverse of A. If A is an orthogonal matrix we have $A^T A = I$, and therefore $A^T = A^{-1}$.

(iii) If A is orthogonal, by (ii) we have $A^T = A^{-1}$, and so $AA^T = I$. Rewrite this as $(A^T)^T A^T = I$. By (i) we deduce that A^T is an orthogonal matrix, i.e. the columns of A^T are an orthonormal set. But the columns of A^T are the rows of A, so the rows of A are an orthonormal set.