MATH 215, SOLUTIONS FROM SEC 5.3

#2. Let U be the set of symmetric n-by-n matrices, and W the set of skew-symmetric matrices. This problem asks you do a number of things.

Prove that U, W are vector spaces: We'll give a detailed proof that U is a vector space; the proof that W is a vector space is similar. To prove that U is a vector space, we have to prove:

- if $A, B \in U$ then $A + B \in U$;
- if $A \in U$ and $c \in \mathbb{R}$, then $cA \in U$.

Recall that a matrix A is symmetric if and only if $A^T = A$. If A, B are symmetric, then

$$(A + B)^T = A^T + B^T = A + B$$

so that $A + B$ is also symmetric; that is, if $A, B \in U$ then $A + B \in U$ as well. If A is symmetric and $c \in \mathbb{R}$, then

$$(cA)^T = c(A^T) = cA\,,$$

so cA is symmetric; that is, if $A \in U$ then $cA \in U$. This proves that U is a vector space. The proof for W is similar (using the fact that A is skew-symmetric if and only if $A^T = -A$).

Prove that $U + W = M(n, \mathbb{R})$: In other words, we need to prove that every n-by-n matrix is the sum of a symmetric matrix and a skew-symmetric matrix. We've already done this, a long time ago, in problems #13 and #14 from section 1.2, but let's remember briefly how this goes. If A is any n-by-n matrix, then

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)\,,$$

and $\frac{1}{2}(A + A^T)$ is symmetric, while $\frac{1}{2}(A - A^T)$ is skew-symmetric.

Compute the dimensions of U, W: First, note that $U \cap W = \{0\}$, since the only matrix that is both symmetric and skew-symmetric is the zero matrix. (If $A^T = A$ and $A^T = -A$, then $A = -A$ and so $A = 0$. This was on your first in-class test.) We have the dimension formula

$$\dim(U) + \dim(W) = \dim(U + W) + \dim(U \cap W) = n^2 + 0 = n^2$$

since $U + W = M(n, \mathbb{R})$ and $\dim(M(n, \mathbb{R})) = n^2$. So the dimensions of U and W sum to n^2. Let's look at what's going on for $n = 2$. Every symmetric 2-by-2 matrix has the form

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

Heuristically, there are three variables, and so we expect that the dimension of this of these matrices is 3. To *prove* this, observe that this means every 2-by-2 symmetric matrix can be written as a linear combination

$$a\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + b\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + c\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}.$$

These three matrices span the space of symmetric 2-by-2 matrices and are obviously linearly independent (the only way to make this linear combination equal to zero is to take $a = b = c = 0$), so they are a basis for the space; therefore the space really does have dimension 3.

Now let's turn to the n-by-n case. The heuristic explanation is the same as in the 2-by-2 case: if you write out a general symmetric matrix, the i, j entry is equal to the j, i entry, so you have one variable for each entry above the diagonal, and one variable for each entry along the main diagonal (and the variables below the diagonal are the same as the ones above the diagonal), e.g. $n = 3$:

$$\begin{pmatrix}a & b & c\\ b & d & e\\ c & e & f\end{pmatrix}.$$

The number of entries along the diagonal is n, so there are $n^2 - n$ entries *not* on the diagonal. Half of these are above the diagonal and half are below, so there are $(n^2 - n)/2$ entries above the diagonal (and the same number below). Thus the total number of variables in a symmetric matrix is the number of variables above the diagonal plus the number of variables along the diagonal, i.e., $(n^2 - n)/2 + n = (n^2 + n)/2$.

We can also give a proof, as before, by giving a basis. If $i < j$, let S_{ij} be the symmetric matrix all of whose entries are 0, except that its i, j entry and j, i entry are both equal to 1. Let S_{ii} be the symmetric matrix all of whose entries are 0, except that its i, i entry is 1. As above, there are $(n^2 - n)/2 + n = (n^2 + n)/2$ of these matrices. The S_{ij} span the space of symmetric matrices, because if $A = (a_{ij})$ is symmetric then

$$A = \sum_{i \le j} a_{ij} S_{ij}.$$

The S_{ij} are clearly linearly independent, because no two of them have a non-zero entry in the same position. Hence they are a basis, and so $\dim(U) = (n^2 + n)/2$.

Now we can use the dimension formula to compute that $\dim(W) = n^2 - \dim(U) = (n^2 - n)/2$. Alternately, you can give a proof for $\dim(W)$ that mirrors the proof above for symmetric matrices, by making a similar heuristic argument and by giving a basis for the space of skew-symmetric matrices. I will leave that to you as a further challenge.

#4. A typical way to prove that two sets A, B are equal is to prove both that $A \subset B$ and $B \subset A$. That is the approach that we will take.

First let us prove that $(U + V) \cap W$ is contained in $U + (V \cap W)$. Let x be an element of $(U + V) \cap W$. In other words: x is an element of $U + V$, and x is also an element of W. We want to show that x is an element of $U + (V \cap W)$.

The fact that x is an element of $U + V$ means that $x = u + v$ for some $u \in U$ and $v \in V$. Since x is in W, this means that $u + v$ is in W. But U is a subset of

W, so u is in W, and therefore $-u$ is in W. Therefore

$$v = (u + v) + (-u)$$

is an element of W. So v is an element of V, and also an element of W, i.e. v is an element of $V \cap W$. Thus $x = u + v$ with $u \in U$ and $v \in V \cap W$, so x is in $U + (V \cap W)$, as desired.

Now let us prove the reverse. Suppose $x \in U + (V \cap W)$. That is, $x = u + y$ with $u \in U$ and $y \in V \cap W$. Since $y \in V$, we see that $x = u + y$ with $u \in U$ and $y \in V$, so $x \in U + V$. Since $u \in U \subset W$ and $y \in V \cap W$, we see that u, y are both in W, and so $x = u + y \in W$. Therefore $x \in U + V$ and $x \in W$, i.e., $x \in (U + V) \cap W$, as desired.

Another way to see the reverse is to note that clearly $U \subset (U + V)$ and $U \subset W$, so $U \subset (U + V) \cap W$; and similarly $V \cap W \subset V \subset U + V$ and $V \cap W \subset W$, so $V \cap W \subset (U + V) \cap W$. Since both U and $V \cap W$ are subspaces of $(U + V) \cap W$, so is $U + (V \cap W)$.

#6. Recall that $P_{14}(\mathbb{R})$ is the space of polynomials of degree at most 13 with real coefficients, so $\dim P_{14}(\mathbb{R}) = 14$. Since $U+W$ is a subspace of $P_{14}(\mathbb{R})$, it follows that $\dim(U + W) \leq 14$. Then

$$\dim(U \cap W) = \dim(U) + \dim(W) - \dim(U+W) = 7+11-\dim(U+W) \geq 7+11-14 = 4\,.$$

To see that equality can occur, we need to find U, W of the correct dimensions, and such that U, W together generate all of $P_{14}(\mathbb{R})$. Here is one example: $U = \langle 1, x, \ldots, x^6 \rangle$ and $W = \langle x^3, x^4, \ldots, x^{13} \rangle$, so that $U \cap W = \langle x^3, x^4, x^5, x^6 \rangle$.