MATH 511B, SELECTED SOLUTIONS FROM HOMEWORKS 8-10

Homework 8

8.7. If G is commutative, so is $k[G]$. Suppose G is cyclic, generated by σ. Since $k[G]$ is commutative we can certainly map $\phi : k[x] \to k[G]$ by fixing k and sending x to σ, and this is evidently surjective. If $G \cong \mathbb{Z}$ then the map is injective as well. On the other hand, if $G \cong \mathbb{Z}/n\mathbb{Z}$ then the kernel of ϕ is an ideal $(f(x))$ in $k[x]$ containing $x^n - 1$, so $f(x)$ divides $x^n - 1$. But the kernel of ϕ cannot contain a polynomial of degree less than n , so $(f(x)) = (x^n - 1)$. (Alternately, the kernel contains $x^n - 1$, so $k[x]/(x^n - 1)$ surjects onto $k[G]$. and this is an isomorphism by a dimension count.)

Homework 9

8.35.(ii) You can show that $A_1(k)$ is left-Noetherian by imitating the proof of the Hilbert basis theorem.

To check that $A_1(k)$ is simple (i.e., that it has no nontrivial proper two-sided ideals), suppose that I is a nonzero two-sided ideal; we will prove that $1 \in I$. To begin, note by induction that we have the more general relation (valid even for $n = 0$):

$$xy^n = y^n x + ny^{n-1}\,,$$

and similarly

$$x^n y = yx^n + nx^{n-1}\,.$$

Suppose $f \in I$ is nonzero and write $f = y^n f_n(x) + \cdots + yf_1(x) + f_0(x)$ with $f_n(x) \neq 0$. Now let us compute $xf - fx$:

$$\begin{aligned} xf &= \sum_{i=0}^{n} xy^i f_i(x) \\ &= \sum_{i=0}^{n} (y^i f_i(x)x + iy^{i-1} f_i(x)) \\ &= fx + \sum_{i=0}^{n-1} (i+1)y^i f_{i+1}(x)\,. \end{aligned}$$

Thus $xf - fx = ny^{n-1}f_n(x) + \cdots + 2yf_2(x) + f_1(x) \in I$, and iterating a total of n times yields $n!f_n(x) \in I$. Now suppose $f_n(x) = a_m x^m + \cdots + a_0$ with $a_m \neq 0$. Since $g(x)y - yg(x) = g'(x)$ for any polynomial $g \in k[x]$, iterating this operation m times on $f_n(x)$ shows $n!m!a_m \in I$. Since k has characteristic zero, we conclude that $1 \in I$.

To show the cancellation laws, it suffices to show that $ab = 0$ in $A_1(k)$ implies $a = 0$ or $b = 0$. The simplest way to check this is probably as follows. Note that each element of $A_1(k)$ is an endomorphism of $k[t]$. One checks that the kernel of a nonzero element of $A_1(k)$ has kernel that is a *finite-dimensional* k-subspace of $k[t]$ (this does take some checking, though). Since $k[t]$ is infinite dimensional over k, the composition of two elements with finite-dimensional kernel cannot equal 0.

Homework 10

8.36. Map $A \to \mathrm{End}_k(A)$ by sending a to the left-multiplication-by-a map. This is a ring homomorphism. Choosing any basis of A over k yields an isomorphism $\mathrm{End}_k(A) \cong M_n(k)$.

8.37.

(iii) Suppose $kG = \mathcal{G} \oplus V$. Then $V \cong kG/\mathcal{G}$, and the latter is already known to be isomorphic to $V_0(k)$. We conclude that V is generated by an element $x \in kG$ such that $gx = x$ for all $x \in G$. This implies x has the form $a\sum_{g\in G} g$, as desired.

(iv) If kG is semisimple, then every submodule of kG is a direct summand, so $kG = \mathcal{G} \oplus V$ for some V. From (iii) we conclude $V = \langle a\sum_{g\in G}\rangle$ for some a. But if $p \mid \#G$ we have $a\sum_{g\in G} \in \mathcal{G}$, and the sum $\mathcal{G} \oplus V$ cannot be direct. This is a contradiction.

8.38. Every simple $M_n(\Delta)$-module is a quotient of $M_n(\Delta)$, hence a Jordan-Hölder factor of $M_n(\Delta)$. Let $COL(\ell)$ denote the ideal consisting of all matrices with zeros outside the ℓth column. It is certainly *not* the case that every minimal ideal in $M_n(\Delta)$ is one of the ideals $COL(\ell)$. However, it is true that

$$M_n(\Delta) = COL(1) \oplus \cdots \oplus COL(n)$$

and $COL(\ell)$ is isomorphic to the simple $M_n(\Delta)$-module Δ^n for each ℓ. By the Jordan-Hölder theorem every simple $M_n(\Delta)$-module must be isomorphic to Δ^n.

8.39. Suppose $a \in R \setminus \{0\}$ and consider the descending chain of *left*-ideals $\langle a^i\rangle$. Since R is left-artinian there exists i such that $\langle a^i\rangle = \langle a^{i+1}\rangle$, and we have u such that $a^i = ua^{i+1}$, i.e., $(1 - ua)a^i = 0$. Since $a \neq 0$ and R has no zero-divisors we conclude $1 = ua$. Thus a is *left*-invertible and u is right-invertible. This in and of itself does not show a is a unit: we must check that it is also right-invertible. But u is a nonzero element of R, so this argument shows that u is left-invertible as well, and u *is* a unit. Hence a is a unit as well.

8.40. To make the statement of part (ii) correct, either assume that k is algebraically closed, or else replace "linear" by "irreducible". To show that $k[T]$ is not semisimple if some $e_p > 1$, the easiest way is to note that $k[x]/(x - a_p)^{e_p}$ has nonzero nilpotent elements (e.g. $x - a_p$), from which you can show that the Jacobson radical of $k[T]$ is nontrivial.