MATH 511A, HOMEWORK 1

Throughout this assignment, let $\mathbb{F}$ be a field.

1. Let V, W be $\mathbb{F}$-vector spaces, and suppose $T : V \to W$ is a linear transformation.

(a) The map T is said to be invertible if there exists a linear transformation $T^{-1} : W \to V$ such that $T^{-1} \circ T$ is the identity on V and $T \circ T^{-1}$ is the identity on W. Prove that T is invertible if and only if it is one-to-one and onto. (That is, it is invertible if and only if it is invertible as a map of sets.)

(b) If T is invertible, show that $\{v_i\}$ is a basis of V if and only if $\{T(v_i)\}$ is a basis of W. Conclude in this case that V and W have the same dimension.

In problems #2-#4, let A be a k-by-n matrix with entries in $\mathbb{F}$, and let A_0 be the reduced row echelon form of A.

2. Prove that the subspace of $\mathbb{F}^k$ spanned by the columns of A has the same dimension as the subspace of $\mathbb{F}^k$ spanned by the columns of A_0. (Consider the effect that a single row operation has on the columns, and use #1(b).)

3. Show that the effect of any single row operation on A can be obtained by multiplying A on the left by an invertible k-by-k matrix. Conclude that there exists an invertible matrix B with $BA = A_0$.

4. Prove that the rows of A are linearly dependent if and only if A_0 has a row consisting entirely of zeros.

5. Let $\mathbb{F}^{\mathbb{N}}$ be the vector space consisting of infinite sequences $(a_1, a_2, \ldots)$ of elements of $\mathbb{F}$, with addition and scalar multiplication defined component by component. As we saw in class, the subspace $\mathbb{F}^{\mathbb{N}}_{\text{fin}}$ has a countable basis, namely the vectors $(1, 0, 0, \ldots)$, $(0, 1, 0, \ldots)$, etc.

Let $v_1, v_2, \ldots$ be a countable sequence of vectors in $\mathbb{F}^{\mathbb{N}}$. Give a proof in the spirit of Cantor's diagonalization argument that the vectors $v_1, v_2, \ldots$ cannot span $\mathbb{F}^{\mathbb{N}}$, and conclude that $\mathbb{F}^{\mathbb{N}}$ has uncountable dimension. This makes precise the notion that $\mathbb{F}^{\mathbb{N}}$ is *much* larger than $\mathbb{F}^{\mathbb{N}}_{\text{fin}}$.