MATH 511A, HW 11: A CERTAIN GROUP OF ORDER 256

In this homework, we will investigate an example (due to Carmichael) of a group of order 256 whose set of commutators is not equal to its commutator subgroup.

Let G be the subgroup of S_{16} generated by the following permutations:

$$\begin{aligned} z_0 &= (1,3)(2,4) \\ z_1 &= (5,7)(6,8) \\ z_2 &= (9,11)(10,12) \\ z_3 &= (13,15)(14,16) \\ y_0 &= (1,3)(5,7)(9,11) \\ y_1 &= (1,2)(3,4)(13,15) \\ y_2 &= (5,6)(7,8)(13,14)(15,16) \\ y_3 &= (9,10)(11,12)\,. \end{aligned}$$

1.

(a) Verify that $z_0, \ldots, z_3$ lie in the center of G.
(b) Check that $z_0, \ldots, z_3$ generate a subgroup of order 16, and deduce that the order of $Z(G)$ is at least 16.

2. For any i, j, verify that the commutator $[y_i, y_j]$ is an element of $\{1, z_0, \ldots, z_3\}$, and that each z_k occurs as such a commutator. Specifically, check that

$$\begin{aligned} [y_0, y_1] &= z_0 \\ [y_0, y_2] &= z_1 \\ [y_0, y_3] &= z_2 \\ [y_1, y_2] &= z_3 \\ [y_1, y_3] &= 1 \\ [y_2, y_3] &= 1\,. \end{aligned}$$

(Be sure to explain why it suffices to check just these six commutators.) Conclude that $G' = \langle z_0, z_1, z_2, z_3 \rangle$, so $\#G' = 16$.

3. Show that $G/\langle z_0, z_1, z_2, z_3 \rangle$ is a commutative group of order at most 16, and argue that it has order exactly 16 if and only if no element of the form $y_0^{\epsilon_0} y_1^{\epsilon_1} y_2^{\epsilon_2} y_3^{\epsilon_3}$ for $\epsilon_i \in \{0, 1\}$ and at least one $\epsilon \neq 0$ lies in $\langle z_0, z_1, z_2, z_3 \rangle$ (in which case these elements form a set of coset representatives for $\langle z_0, z_1, z_2, z_3 \rangle$).

4. Consider two elements of the form $y_0^{\epsilon_0} y_1^{\epsilon_1} y_2^{\epsilon_2} y_3^{\epsilon_3}$, $y_0^{\delta_0} y_1^{\delta_1} y_2^{\delta_2} y_3^{\delta_3}$ for $\epsilon_i, \delta_i \in \{0, 1\}$.

(a) Prove that
$$(y_0^{\epsilon_0} y_1^{\epsilon_1} y_2^{\epsilon_2} y_3^{\epsilon_3}) \cdot (y_0^{\delta_0} y_1^{\delta_1} y_2^{\delta_2} y_3^{\delta_3})$$
is equal to
$$z_0^{\delta_0 \epsilon_1} z_1^{\delta_0 \epsilon_2} z_2^{\delta_0 \epsilon_3} z_3^{\delta_1 \epsilon_2} (y_0^{\delta_0+\epsilon_0} y_1^{\delta_1+\epsilon_1} y_2^{\delta_2+\epsilon_2} y_3^{\delta_3+\epsilon_3}) .$$

(b) Use the result of part (a) to prove the formula
$$[y_0^{\epsilon_0} y_1^{\epsilon_1} y_2^{\epsilon_2} y_3^{\epsilon_3}, y_0^{\delta_0} y_1^{\delta_1} y_2^{\delta_2} y_3^{\delta_3}] = z_0^{\delta_0 \epsilon_1 - \delta_1 \epsilon_0} z_1^{\delta_0 \epsilon_2 - \delta_2 \epsilon_0} z_2^{\delta_0 \epsilon_3 - \delta_3 \epsilon_0} z_3^{\delta_1 \epsilon_2 - \delta_2 \epsilon_1} .$$

(c) From the formula in part (b), prove that if $y_0^{\epsilon_0} y_1^{\epsilon_1} y_2^{\epsilon_2} y_3^{\epsilon_3}$ lies in $Z(G)$, then $\epsilon_0 = \epsilon_1 = \epsilon_2 = \epsilon_3 = 0$.

5. Conclude from the previous problems that

(i) $Z(G) = \langle z_0, z_1, z_2, z_3 \rangle$, so $G' = Z(G)$ and $\#Z(G) = 16$;
(ii) $\#(G/Z(G)) = 16$;
(iii) $\#G = 256$.

6. Use the formula in 4(b) to prove that $z_2 z_3$ is *not* the commutator of two elements of G. Hence $z_2 z_3$ is an example of an element which is contained in the commutator subgroup but is not itself a commutator.