

MATH 511A, HOMEWORK 2

Throughout this assignment, let \mathbb{F} be a field and let V, W be \mathbb{F} -vector spaces.

1. Suppose V, W are finite dimensional, and let v_1, \dots, v_n and v'_1, \dots, v'_n be two bases for V . Similarly let w_1, \dots, w_k and w'_1, \dots, w'_k be two bases for W .

Let $f : V \rightarrow W$ be a linear transformation, and suppose f has matrix A with respect to the bases v_1, \dots, v_n and w_1, \dots, w_k , and matrix A' with respect to the bases v'_1, \dots, v'_n and w'_1, \dots, w'_k .

- (a) How is the matrix A' related to the matrix A ?
- (b) If $V = W$, so that $f : V \rightarrow V$, then we can use the same basis for both copies of V ; that is, we can let A be the matrix with respect to the basis v_1, \dots, v_n for both copies of V , and A' the matrix with respect to the basis v'_1, \dots, v'_n for both copies of V . How is A' related to A in this special case?

2. Prove that there is a linear transformation $T : \mathbb{F}^{\mathbb{N}} \rightarrow (\mathbb{F}_{\text{fin}}^{\mathbb{N}})^*$ sending (a_i) to the map $\mathbb{F}_{\text{fin}}^{\mathbb{N}} \rightarrow \mathbb{F}$ that sends

$$(b_i) \mapsto \sum a_i b_i.$$

Show that T is an isomorphism, and conclude that $\mathbb{F}_{\text{fin}}^{\mathbb{N}}$ is *not* isomorphic to its dual.

3. Let G be a commutative group with identity element 0 , and let H be a subgroup of G . Prove that there exists a subgroup K of G that is maximal with respect to the property that $H \cap K = \{0\}$.

4. Let V, W be vector spaces, and let $f : W \rightarrow W'$ be a linear transformation. Prove that composition with f gives a linear transformation

$$f_* : \text{Hom}_{\mathbb{F}}(V, W) \rightarrow \text{Hom}_{\mathbb{F}}(V, W').$$

Similarly, if $g : V \rightarrow V'$ is a linear transformation, prove that (pre)composition with g gives a linear transformation

$$g^* : \text{Hom}_{\mathbb{F}}(V', W) \rightarrow \text{Hom}_{\mathbb{F}}(V, W).$$

5. For the final problem, we need the notion of a *short exact sequence* of vector spaces. If $f : V' \rightarrow V$ and $g : V \rightarrow V''$ are two linear transformations, then the sequence of maps $V' \xrightarrow{f} V \xrightarrow{g} V''$ is said to be *exact at V* if $\text{im}(f) = \text{ker}(g)$. The sequence of maps

$$(\star) \quad 0 \rightarrow V' \xrightarrow{f} V \xrightarrow{g} V'' \rightarrow 0$$

is a *short exact sequence* if it is exact at all of V', V , and V'' .

- (a) Show that (\star) is a short exact sequence if and only if
 - f is injective,

- $\text{im}(f) = \ker(g)$, and
 - g is surjective.
- (b) If (\star) is a short exact sequence, show that $g \circ f = 0$.
- (c) If (\star) is a short exact sequence, prove that

$$0 \rightarrow \text{Hom}_{\mathbb{F}}(V'', W) \xrightarrow{g^*} \text{Hom}_{\mathbb{F}}(V, W) \xrightarrow{f^*} \text{Hom}_{\mathbb{F}}(V', W) \rightarrow 0$$

is also a short exact sequence.