MATH 511A, HOMEWORK 3

1. *Restriction of scalars.* Let V be a finite-dimensional vector space over $\mathbb{C}$. We can obtain a vector space $V_{\mathbb{R}}$ over $\mathbb{R}$ by restricting scalar multiplication from $\mathbb{C}$ to $\mathbb{R}$: that is, the vectors of $V_{\mathbb{R}}$ are the vectors of V, with the same addition and scalar multiplication operations, except that we allow scalar multiplication only by elements of $\mathbb{R}$. Determine $\dim_{\mathbb{R}} V_{\mathbb{R}}$ in terms of $\dim_{\mathbb{C}} V$.

2. Let $S \subset M_n(\mathbb{F})$ be the set of symmetric n-by-n matrices over $\mathbb{F}$, and let $T \subset M_n(\mathbb{F})$ be the set of skew-symmetric n-by-n matrices over $\mathbb{F}$. That is, $S = \{A \in M_n(\mathbb{F}) : A^t = A\}$, while $T = \{A \in M_n(\mathbb{F}) : A^t = -A\}$, where the superscript t denotes transpose. Prove that S, T are subspaces of $M_n(\mathbb{F})$ and give a necessary and sufficient condition on the field $\mathbb{F}$ so that $M_n(\mathbb{F})$ is the internal direct sum of S and T.

3. Let V be an $\mathbb{F}$-vector space and $V_1, \ldots, V_n \subsetneq V$ a finite collection of proper subspaces of V. Suppose that $V = \cup_i V_i$. Prove that $\mathbb{F}$ is finite.

4. Let $\mathbb{F}[x]$ be the space of polynomials with coefficients in $\mathbb{F}$. Let $D : \mathbb{F}[x] \to \mathbb{F}[x]$ be the algebraic *differentiation* map, the unique linear map such that $D(x^n) = nx^{n-1}$. (Note that $x^0, x^1, x^2 \ldots$ are a basis of $\mathbb{F}[x]$ as an $\mathbb{F}$-vector space.) Determine $\ker(D)$ and $\operatorname{im}(D)$. Your answer will depend on the characteristic of $\mathbb{F}$.

5. Read sections I.4 (permutations and signs) and II.7 (determinants) and make sure you understand them.